

General Nonlocality in Quantum Fields

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The natural recognition of quantum nonlocality follows from the fact that a quantum wave is spatially extended. The waves of fermions display nonlocality in low energy limit of quantum fields. In this *ab initio* paper we propose a complex-geometry model that reveals the affection of nonlocality on the interaction between material particles of spin-1/2. To make nonlocal properties appropriately involved in a quantum theory, the special unitary group $SU(n)$ and spinor representation $D^{(1/2,1/2)}$ of Lorentz group are generalized by making complex spaces—which are spanned by wave functions of quantum particles—curved. The curved spaces are described by the geometry used in General Relativity by replacing the real space with complex space and additionally imposing the analytic condition on the space. The field equations for fermions and for bosons are respectively associated with geodesic motion equations and with local curvature of the considered space. The equation for fermions can restore all the terms of quadratic form of Dirac equation. According to the field equation it is found that, for the $U(1)$ field [generalized Quantum Electrodynamics (QED)], when the electromagnetic fields \vec{E} and \vec{B} satisfy $\vec{E}^2 - \vec{B}^2 \neq 0$, the bosons will gain masses. In this model, a physical region is empirically defined, which can be characterized by a determinant occurring in boson field equation. Applying the field equation to $U(3)$ field [generalized Quantum Chromodynamics (QCD)], the quark-confining property can be understood by carrying out the boundary of physical region. And it is also found out that under the conventional form of interaction vertex, $\gamma_\mu A^\mu$, only when the colour group $SU(3)$ is generalized to $U(3)$ is it possible to understand the strongly bound states of quarks.

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I. INTRODUCTION

Nonlocality is an important phenomenon in nature, particularly in quantum world. The direct recognition of quantum nonlocality comes from the fact that a quantum wave is spatially extended, in contrast to the point-model for classical particles. In this paper we mainly discuss how the nonlocality affects the interactions between material particles of spin-1/2. The problem is intriguing since the nonlocality has been gleamingly implied by the renormalization of conventional Quantum Field Theory (CQFT), whence most relevant calculations have to be regulated by momentum cutoff to contain the non-point effect. The technique however, is usually available only at high-energy scale, the case where the wavelengths of particles are ultra short. Here we take into account the nonlocal effect emerging within the range of interactions—possibly a few wavelengths; but we don't get involved in the hotly discussed long-distance effects relating to entangled states and their applications—such as quantum information, quantum communication and quantum computation etc..

Up to date, we have recognized that one cannot accurately measure the spatial coordinates of a proton by making an accelerated electron as probe, unless its wavelength is much shorter than the "diameter" of the proton. But the proton would be smashed and some other hadrons will be involved in the final state (and thus the scattering becomes inelastic) if making the electron's wavelength short enough. In the case of elastic scattering, the detected proton becomes a **singularity** for the electron's wave. The reason may be that, in the measurements, the quantity (coordinates) we inquire is not at the same spatial level as that the quantum entities settled in—the coordinate is a four-dimension quantity but the electron's or proton's wave is eight-dimension, or put it in mathematical terminology, the quantity we inquire is real but a quantum object is complex.

It is concluded from purely mathematical point of view that, only located in a space with dimension equal to or larger than that of the detected objects can an observer get complete information of direct measurement. As a tentative method and a starting point, in this paper we propose an **equal observer**, e.g. an electron, is also put into the Hilbert space to observe another electron or other fermions such as protons. Presumably, they are at the same spatial level. Therefore the electron can use the metric (gauge) appropriate for the observed objects to measure physical observables. The method of **equal observer** is conducive to describing the observed quantum wave (**nonlocal entity**) as a whole with possibly less interaction-information lost, unlike in conventional quantum mechanics (CQM) where quantum wave is expressed on the basis of space-time points. The dynamics for the equal observer of a quantum wave is believed to be different from CQM.

In this paper we employ the similarity between quantum **singularity** and gravitational **singularity** to describe how one fermion observes (interacts with) another fermion, and dynamically apply the formalism of General Relativity (GR) by generalizing its space from real to complex [Fig. 1]. As for the elastic scattering of electron and proton, in

calculating the radiative corrections to the lowest order of scattering process by employing Quantum Electrodynamics (QED), we encounter the divergence rooted from leading-order potential form $\frac{1}{r}$ while making momentum $|\vec{p}| \rightarrow \infty$. In calculating the collision of two heavy celestial bodies by using GR, the similar singularity rooted also from the form $\frac{1}{r}$ is encountered, but there the puzzle of divergence is automatically circumvented by carrying out a horizon, the outer of which is physical region, and the inner of which, now known as black hole region, is unphysical. Quantum mechanically, the nonlocal region is usually unobservable in our space-time, and thus unphysical. Enlightened by such physical scenario, we expect to define physical region for elemental fermions in complex space.

In analogy to GR, the principle of nonlocality for two interacting fermions is: **There always exists a complex frame for observer (one fermion) in which the observed fermion (another fermion) looks like a plane wave, no matter the existence of interaction.**

CQFT itself can also lead us to perceive the implicit and profound relationship between nonlocality (quantum wave) and complex-curvature. Generally, we interpret the scattering matrix between initial state $|i\rangle$ and final state $|f\rangle$ as $S_{fi} = \langle f | \infty \rangle = \langle f | S | i \rangle$, where $|i\rangle = |-\infty\rangle$ can be any state of a complete set. In this formalism, the operator S (or alternatively, the Hamiltonian) is assumed known. Then the matrix elements S_{fi} —whose square is proportional to transition rate between initial and final states—can be evaluated. Whereas from an equal-observer angle, all the states $|i\rangle$ are known and the state $|\infty\rangle_f$ can be observed, so the operator S can be carried out formally $S_f = \sum_i | \infty \rangle_f \langle i |$, consequently the interaction becomes known. This latter opposite thought reminds us of the physics in GR, where the force can be realized by the curving of space-time. So, if only the S -matrix is defined locally in complex-space (a quantum wave is viewed as a point in such space, and as a whole), the differential geometry for **nonlocal entity** would definitely occur. [Note: For convenience, in what follows we will not employ the language of S -matrix, though relevant.] The further understanding of the relationship between nonlocality and curvature is achieved in section 10, where the local conservation laws don't exist. In summary, one will gradually be aware of that all of the above intuitive knowledge on nonlocality could be *transferred* to and *interpreted* by the Complex Geometry we used, *via* the motion equation for fermions and the field equation for bosons in section 8~9.

The main results of the paper are as follows: From our dynamical motion equation we can restore all the terms appearing in the quadratic form of Dirac equation. For the $U(1)$ field [generalized QED], if electromagnetic fields \vec{B} and \vec{E} satisfy $\vec{E}^2 - \vec{B}^2 \neq 0$, the bosons will gain masses. Based on the discussion of physical region, we can attain two qualitative understandings of quark confinement.

In order not to make readers confused by mathematical details, we only list some main results of the real and complex geometry with detailed explanations and the necessary calculating techniques, then immediately apply them to physics problems—discussion of dynamical equations and the definition of physical region. The remainder of this paper is arranged as follows. In sections 2-5 we introduce the necessary calculating techniques of geometry, the readers familiar with the contents can neglect this part and directly turn to section 6. Then in sections 6-9 we apply them to construct the motion equations for fermions and field equation for bosons. The conservation laws are sketched in section 10. In section 11, the physical region is defined for $U(1)$ field [generalized QED] and $U(3)$ field [generalized QCD]. Consequently the qualitative understandings to confinement of quarks is presented under the approximation of interaction vertex $\gamma_\mu A^\mu$. Finally a concluding section is presented to summarize the paper and give some remarks on the applicability of the theory. In appendix A, the derivation of motion equation for fermions is elaborated.

II. A USEFUL TOOL: EXTERIOR DIFFERENTIATION d

Exterior product proves to be a powerful tool in calculating tensors. It obeys the rules of Grassman algebra. One of its usages is to give rise to new tensors of higher order. For example, let $\{x_1, x_2, \dots, x_n\}$ be of coordinates for a space, and $f(x_1, x_2, \dots, x_n)$ an arbitrary scalar function with respect to these coordinates, then the total differential of the function f is $df = \sum_i \frac{\partial f}{\partial x_i} dx_i, i = 1, 2, \dots, n$. We call $(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n})_p$ a one-order tensor, or a vector, and $(dx_1, dx_2, \dots, dx_n)_p$ the basis in the neighborhood of a given point p . In terms of exterior product the df is called differential 1-form. Now we can construct 2-form on the basis of the 1-form: $ddf = \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j, i = 1, 2, \dots, n$. The sign \wedge denotes the exterior product. It satisfies the antisymmetric rule: $dx_i \wedge dx_j = -dx_j \wedge dx_i$, according to which one can easily conclude $ddf = 0$. Conventionally, an n -form is written as

$$T_{ij\dots k} dx^i \wedge dx^j \wedge \dots \wedge dx^k. \quad (2.1)$$

The repeated index henceforth in this whole paper means summation. Here the $T_{ij\dots k}$ are components of the k -order tensor, and the exterior products $dx^i \wedge dx^j \wedge \dots \wedge dx^k$ are the bases.

The above forms can be readily extended to complex variables, e.g., the two complex variables $\{z, \bar{z}\}$. And the only 2-form of these complex variables is $g(z, \bar{z}) dz \wedge d\bar{z}$ where $g(z, \bar{z})$ is a function of z and \bar{z} . Here z and \bar{z} are independent variables.

III. CAUCHY-RIEMANN CONDITION FOR COMPLEX MANIFOLD

Usually a space is called n -dimension manifold if it can be formed as smoothly as possible by affixing many infinitesimally flat patches of \mathbf{R}^n (n -dimension real space), \mathbf{C}^n (n -dimension complex space), or other sorts of n -dimension spaces. In fact in this paper we are only concerned about a particular sort of differential manifolds, but for convenience we will call it manifold without specification.

Above all, an important property for a given space is the existence of derivatives, which is essential to build up a description of differential geometry. As for Hilbert space, the existence of derivatives may be determined by that of wave function $\psi(x)$ in it. In general, the existence of derivatives of a real function can be understood intuitively. However, for a complex function, that demands the Cauchy-Riemann condition be satisfied. When the complex derivative can be defined "everywhere," the function is said to be **analytic**. Now we review it in respect of complex geometry.

Conventionally, we identify an n -dimension complex manifold with a $2n$ -dimension real manifold. The complex coordinates and real coordinates have the relation

$$z^\alpha = x^\alpha + iy^\alpha, \bar{z}^\alpha = x^\alpha - iy^\alpha, \alpha = 1, \dots, n, \quad (3.1)$$

then the following 1-form is written straightforwardly

$$dz^\alpha = dx^\alpha + idy^\alpha, d\bar{z}^\alpha = dx^\alpha - idy^\alpha, \quad (3.2)$$

and subsequently

$$\frac{\partial}{\partial x^\alpha} = \frac{\partial z^\alpha}{\partial x^\alpha} \frac{\partial}{\partial z^\alpha} + \frac{\partial \bar{z}^\alpha}{\partial x^\alpha} \frac{\partial}{\partial \bar{z}^\alpha} = \frac{\partial}{\partial z^\alpha} + \frac{\partial}{\partial \bar{z}^\alpha}, \quad (3.3 \text{ a})$$

$$\frac{\partial}{\partial y^\alpha} = \frac{\partial z^\alpha}{\partial y^\alpha} \frac{\partial}{\partial z^\alpha} + \frac{\partial \bar{z}^\alpha}{\partial y^\alpha} \frac{\partial}{\partial \bar{z}^\alpha} = i \frac{\partial}{\partial z^\alpha} - i \frac{\partial}{\partial \bar{z}^\alpha}. \quad (3.3 \text{ b})$$

The reverse of the above equations yields

$$\frac{\partial}{\partial z^\alpha} \stackrel{d}{=} \partial_\alpha = \frac{1}{2} \left(\frac{\partial}{\partial x^\alpha} - i \frac{\partial}{\partial y^\alpha} \right), \quad (3.4 \text{ a})$$

$$\frac{\partial}{\partial \bar{z}^\alpha} \stackrel{d}{=} \bar{\partial}_\alpha = \frac{1}{2} \left(\frac{\partial}{\partial x^\alpha} + i \frac{\partial}{\partial y^\alpha} \right), \quad (3.4 \text{ b})$$

here the sign d above the equals sign means a **definition**. Perform exterior-product upon an arbitrary complex scalar function $f(z^\alpha, \bar{z}^\alpha)$, it results in a 1-form

$$df = \partial_\alpha f dz^\alpha + \bar{\partial}_\alpha f d\bar{z}^\alpha \stackrel{d}{=} \partial f + \bar{\partial} f. \quad (3.5)$$

Make $f = u + iv$, we note that $\bar{\partial} f = 0$ is just the Cauchy-Riemann relation

$$\frac{\partial u}{\partial x^\alpha} = \frac{\partial v}{\partial y^\alpha}, \frac{\partial u}{\partial y^\alpha} = -\frac{\partial v}{\partial x^\alpha} \quad (3.6)$$

Another important property of a given space is its transformation. Commonly, the vector fields spanned by basis $(dx_1, dx_2, \dots, dx_n)$ [real space] or by basis $\{(dz^\alpha, d\bar{z}^\alpha), \alpha = 1, \dots, n\}$ [complex space] on manifolds are the objects responsible for addressing the geometry of the manifolds. How do these fields transform from one point to another infinitesimal neighboring point of the manifolds? The transformations are usually constrained by some groups as done in the well-known classical mechanics or quantum fields. We will return to this topic later. Now let's turn to the affection of Cauchy-Riemann relation on a vector basis. Ignoring some unnecessary details, we assume that the transformation of the n -dimension complex manifold has the general form

$$z^\alpha \rightarrow z'^\alpha = f^\alpha(z^1, \bar{z}^1; z^2, \bar{z}^2; \dots; z^n, \bar{z}^n). \quad (3.7)$$

As an application of Eq.(3.6), replacing the function f by f^α , and writing the f^α of Eq.(3.7) explicitly as $f^\alpha = u^\alpha + iv^\alpha$, the form of the Eq.(3.6) now reads

$$\frac{\partial u^\alpha}{\partial x^\beta} = \frac{\partial v^\alpha}{\partial y^\beta}, \quad \frac{\partial u^\alpha}{\partial y^\beta} = -\frac{\partial v^\alpha}{\partial x^\beta}, \quad \alpha, \beta = 1, \dots, n. \quad (3.8)$$

Make $y^\alpha = x^{n+\alpha}$, $v^\alpha = u^{n+\alpha}$ and introduce a $2n \times 2n$ matrix J

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad (3.9)$$

which satisfies $J^2 = -I_{2n}$, I_{2n} denoting the identity matrix of rank $2n$, then the Cauchy-Riemann relation (3.8) can be written

$$J_k^j \frac{\partial u^k}{\partial x^l} = \frac{\partial u^j}{\partial x^k} J_l^k, \quad j, k, l = 1, \dots, 2n, \quad (3.10)$$

and correspondingly an identical relation holds

$$J_l^j = \frac{\partial u^j}{\partial x^k} J_n^k \frac{\partial x^n}{\partial u^l}. \quad (3.11)$$

Write out the Eq. (3.10) alternatively as

$$J_l^k \frac{\partial u^j}{\partial x^k} = \frac{\partial (J_k^j u^k)}{\partial x^l}, \quad (3.12)$$

and regardless of J on u^k , we obtain

$$J \frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial y^\alpha}, \quad J \frac{\partial}{\partial y^\alpha} = -\frac{\partial}{\partial x^\alpha}. \quad (3.13)$$

Combining it with the equation (3.4) gives

$$J \frac{\partial}{\partial z^\alpha} = -i \frac{\partial}{\partial z^\alpha}, \quad (3.14 \text{ a})$$

$$J \frac{\partial}{\partial \bar{z}^\alpha} = i \frac{\partial}{\partial \bar{z}^\alpha}. \quad (3.14 \text{ b})$$

Consistently the results (3.13)~(3.14) can also be expressed using the basis $(dx_1, \dots, dx_n; dy_1, \dots, dy_n)$ and $\{dz^\alpha, d\bar{z}^\alpha\}$, respectively as

$$J dx^\alpha = -dy^\alpha, \quad J dy^\alpha = dx^\alpha, \quad (3.15)$$

and

$$J dz^\alpha = -dy^\alpha + i dx^\alpha = i dz^\alpha, \quad (3.16 \text{ a})$$

$$J d\bar{z}^\alpha = -dy^\alpha - i dx^\alpha = -i d\bar{z}^\alpha. \quad (3.16 \text{ b})$$

From Eq. (3.16), we note that the dz^α and $d\bar{z}^\alpha$ are eigen forms of the matrix J , and the relations are independent of the choice of the coordinates. In this respect J is called the **complex structure** of the complex manifold.

A manifold with complex structure is called **almost** complex manifold.

The above mathematical formulae will become more meaningful if we replace z^α and \bar{z}^α by the wave function ψ^α and $\bar{\psi}^\alpha$ from the Hilbert space (Manifold), which will be recognized in following sections. For convenience, henceforth we will put ψ^α in a more general space, with Hilbert space only as a special case.

IV. A ROUTE TO CONSTRUCT DIFFERENTIAL GEOMETRY FOR A SPACE

Before giving the general definition of metric (gauge), it is helpful to recall its definition in three-dimensional (3-D) space. Using the Cartan Method of Moving Frames [1], let every point $\vec{x} = (x_1, x_2, x_3)$ in 3-D space correspond to a frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ as following. If the point \vec{x} varies with a curvilinear net defined by (u_1, u_2, u_3) , i.e., $x_i = x_i(u_1, u_2, u_3)$, the total differential of \vec{x} then takes the form

$$d\vec{x} = \frac{\partial \vec{x}}{\partial u_i} du_i, \quad i = 1, 2, 3, \quad (4.1)$$

if setting $\frac{\partial \vec{x}}{\partial u_i} = P^{ij} \vec{e}_j$ and $\omega^j = du_i P^{ji}$, then from Eq.(4.1) the point \vec{x} relates to frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ as follows

$$d\vec{x} = \omega^i \vec{e}_i. \quad (4.2)$$

Now define metric tensor as

$$g_{ij} = (\vec{e}_i, \vec{e}_j), \quad (4.3)$$

the parenthesis denotes the inner product of common sense. Combining these definitions, the metric of $d\vec{x}$ reads

$$ds^2 = (d\vec{x}, d\vec{x}) = (\omega^i \vec{e}_i, \omega^j \vec{e}_j) = g_{ij} \omega^i \omega^j. \quad (4.4)$$

From Eq.(4.2), the differential of basis \vec{e}_i can be formally written as

$$d\vec{e}_i = \omega_i^j \vec{e}_j, \quad (4.5)$$

where ω_i^j is also a 1-form similar to ω^i , then the differential for g_{ij} can be calculated as follows

$$dg_{ij} = (d\vec{e}_i, \vec{e}_j) + (\vec{e}_i, d\vec{e}_j) = g_{kj} \omega_i^k + g_{ik} \omega_j^k. \quad (4.6)$$

The above general forms is suitable for any frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Now let's turn to some special cases relevant to concrete frames.

A well-known metric form for coordinates frames is the orthogonal one, i.e. $g_{ij} = (\vec{e}_i, \vec{e}_j) = \delta_{ij}$. With this relation, Eq.(4.4) becomes

$$ds^2 = \omega^i \omega_i, \quad (4.7)$$

and Eqs. (4.6), (4.7) give rise to

$$\omega_i^k = -\omega_k^i. \quad (4.8)$$

Another general/important frame is the **natural frame**, in which \vec{e}_i is defined by

$$\vec{e}_i = \frac{\partial \vec{x}}{\partial u^i}, \quad (4.9)$$

then (4.1) is directly written as

$$d\vec{x} = du^i \vec{e}_i, \quad (4.10)$$

and (4.4) reads

$$ds^2 = g_{ij} du^i du^j. \quad (4.11)$$

The natural frame is of elicitation in studying manifold, which can help us extend the above discussion to n-dimensional space. The space continuously generated by curvilinear net $\{u^1, u^2, \dots, u^n\}$ is a manifold. Now let's turn to its natural bases $\{\vec{e}_i = \frac{\partial \vec{x}}{\partial u^i}\}$.

According to the terminology of quantum mechanics, \vec{e}_i is made by two parts: a matrix (operator) $\frac{\partial}{\partial u^i}$ and a vector \vec{x} . Since the operator parts $\frac{\partial}{\partial u^i}$ are the same for different points \vec{x} , we usually write $\vec{e}_i = \frac{\partial}{\partial u^i}$ for general cases while $\vec{e}_i = \frac{\partial}{\partial u^i} |_{\vec{x}} (= \frac{\partial \vec{x}}{\partial u^i})$ for a given point \vec{x} . In this sense, the general form of inner product g_{ij} becomes

$$g_{ij} = \left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right). \quad (4.12)$$

For a given point p on manifold, $g_{ij}(p)$ gives rise to a mapping that maps two arguments $\frac{\partial}{\partial u_i} |_p$ and $\frac{\partial}{\partial u_j} |_p$ to a real or complex number.

Then what is the relationship between the basis du^i — which has been mentioned in section II—and this $\frac{\partial}{\partial u_i}$? That can be recognized by the integral $\int du^i \frac{\partial}{\partial u^i} f(u) = f(u)$ where f is an arbitrary function, the integrand $f(u)$ is unaltered after integration. So it is reasonable to write

$$(du^i, \frac{\partial}{\partial u^j}) = \delta_j^i . \quad (4.13)$$

The contravariant metric tensor g^{ij} is thus defined by

$$g^{ij} = (du^i, du^j) , \quad (4.14)$$

with $g^{ij}g_{jk} = \delta_k^i$. The above arguments equally hold in complex space.

For a given space, there may exist several ways to define its metric form, *i.e.*, the gauge to measure the space. In the physical respect, we are only interested in the forms like Eq. (4.14), even for complex manifold.

The construction of metrics includes another important feature, *i.e.* the constructed metric ds^2 should be of geometrically invariant quantity with respect to the transformation between frames of the different points of the manifold/space. For a given space, there may exist many ways to jump from one point to another point in its infinitesimal neighborhood. To address the manner of the jump is to determine the transformation group between the corresponding frames. The transformation group is also a key element to define manifold, different groups correspond to different manifolds.

Now let's turn to the general form of differential geometry. Perform the infinitesimal group transformation upon the vector basis $\frac{\partial}{\partial x_i} = e_i$, the change of e_i is then analogous to Eq. (4.5)

$$de_j = \Gamma_j^i e_i , \quad (4.15)$$

Γ_α^β is 1-form (the similar meaning as the above ω_i^j) on the manifold. Conventionally, we take the explicit form $\Gamma_i^j = \Gamma_{ki}^j dx^k$, in which Γ_{ik}^j (not a tensor) is known as **connection**, referring to the manner how to affix the infinitesimal flat-space-patches to form a larger curved space. According to above definition Eq. (4.15), the variation of a vector $X = \xi^i e_i$ under the transformation yields

$$dX = d\xi^i e_i + \xi^i de_i = (d\xi^i + \xi^j \Gamma_j^i) e_i = (D\xi^i) e_i , \quad (4.16)$$

where D is known as covariant differential. The above expressions indicate that the notations de_α , De_α and Γ_α^β are identical: $De_\alpha = \Gamma_\alpha^\beta e_\beta$, which leads us to a conventional denotation $D = \Gamma_\alpha^\beta$ where D and Γ_α^β are both viewed as operators. If for all components $D\xi^i = 0$ is met, the differential dX is called the parallel displacement for vector X , and the path along which the displacement would occur is solvable from the equation $D\xi^i = 0$.

The connection Γ_α^β can be carried out by demanding that it should preserve the inner product of two vector X and Y

$$d(X, Y) = 0 , \quad (4.17)$$

if they are parallelly displaced, *i.e.* $dX = dY = 0$. In the above equation

$$(X, Y) = (\xi^i e_i, \eta^j e_j) = g_{ij} \xi^i \eta^j , \quad (4.18)$$

and ξ^i and η^j are 1-form. We see that

$$0 = d(X, Y) = dg_{ij} \xi^i \eta^j + g_{ij} d\xi^i \eta^j + g_{ij} \xi^i d\eta^j \quad (4.19)$$

leads to

$$(dg_{ij} - g_{kj} \Gamma_i^k - g_{ik} \Gamma_j^k) \xi^i \eta^j = 0 , \quad (4.20)$$

i.e.

$$Dg_{ij} = dg_{ij} - g_{kj} \Gamma_i^k - g_{ik} \Gamma_j^k = 0 . \quad (4.21)$$

The parallel displacement $D\xi^i = 0$ and the conclusion that the connection preserves the metric (X, Y) , *i.e.* Eq. (4.21), are equivalent—they can bring out each other. Note that all the components of Γ_j^i form a matrix $\Gamma = (\Gamma_j^i)$, and likewise components of g_{ij} form a matrix G , Eq. (4.21) can be written in terms of matrices

$$dG = \Gamma^t \cdot G + G \cdot \Gamma , \quad (4.22)$$

Γ^t means the transpose of the matrix Γ . The relation Eq. (4.22) puts a way—which we will elucidate later—to carry out the connection Γ_{jk}^i with respect to the metric g_{ij} .

Let's turn to the most important quantity of manifold—the curvature tensor—which is an invariant 2-form under a certain transformation group that specifies the manifold. Conventionally, the curvature matrix is defined by

$$\Omega = (\Omega_\alpha^\beta) = d\Gamma - \Gamma \wedge \Gamma . \quad (4.23)$$

Taking $\Omega_\alpha^\beta = R_{\alpha j k}^\beta dx^j \wedge dx^k$ and bearing in mind that $\Gamma_\alpha^\beta = \Gamma_{k\alpha}^\beta dx^k$, we can express it explicitly

$$R_{\alpha j k}^\beta = \Gamma_{k\alpha, j}^\beta - \Gamma_{j\alpha, k}^\beta + \Gamma_{j\gamma}^\beta \Gamma_{k\alpha}^\gamma + \Gamma_{k\gamma}^\beta \Gamma_{j\alpha}^\gamma , \quad (4.24)$$

the denotation $,j$ in subscripts stands for the derivative with respect to the j -th variable. Then by $\text{Tr}\Omega$, i.e. making $\alpha = \beta$ in (4.24) [summation convention is applied too.] one gets the Ricci tensor

$$R_{jk} = R_{\alpha j k}^\alpha . \quad (4.25)$$

Above formulae pertain to all manifolds with the metric form like Eq. (4.14).

V. THREE USEFUL COMPLEX MANIFOLDS

A. Connections for almost complex manifold

The analytic property of complex manifold is determined completely by the complex structure, so any connection possessed by complex manifold is demanded to preserve the complex structure. i.e.

$$J D_\alpha = D_\alpha J , \quad (5.1)$$

where D_α means performing differential on a given basis e_α , i.e. $D_\alpha e_\beta = \Gamma_{\alpha\beta}^\gamma e_\gamma$. Apply Eq. (5.1) to Eq. (4.15), remembering the Eq.(3.14) [Please note: for a complex manifold, we use denotations $e_\alpha = \frac{\partial}{\partial z^\alpha}$ and $e_{\bar{\alpha}} = \frac{\partial}{\partial \bar{z}^\alpha}$ hereafter], then

$$J D_\alpha e_\beta = J \Gamma_{\alpha\beta}^\gamma e_\gamma = \Gamma_{\alpha\beta}^\gamma J e_\gamma = -i \Gamma_{\alpha\beta}^\gamma e_\gamma = D_\alpha J e_\beta . \quad (5.2)$$

However, if we apply Eq.(5.1) to $e_{\bar{\alpha}}$, the result becomes

$$J D_\alpha e_{\bar{\beta}} = J \Gamma_{\alpha\bar{\beta}}^\gamma e_\gamma = \Gamma_{\alpha\bar{\beta}}^\gamma J e_\gamma = -i \Gamma_{\alpha\bar{\beta}}^\gamma e_\gamma = -D_\alpha J e_{\bar{\beta}} , \quad (5.3)$$

which violates Eq.(5.1). Therefore, Eq.(5.1) holds only if

$$\Gamma_{\alpha\bar{\beta}}^\gamma \equiv 0 . \quad (5.4 \text{ a})$$

Replace $\Gamma_{\alpha\bar{\beta}}^\gamma$ by $\bar{\Gamma}_{\alpha\bar{\beta}}^\gamma$ and repeat the same procedure, we have

$$\bar{\Gamma}_{\alpha\bar{\beta}}^\gamma \equiv 0 . \quad (5.4 \text{ b})$$

Eqs.(5.4) suggest only the 1-form Γ_α^β and $\bar{\Gamma}_{\bar{\alpha}}^{\bar{\beta}}$ are permitted in the complex manifold. If the connection of a manifold preserves the complex structure J , then the manifold is called almost complex manifold, this statement is equivalent to aforementioned definition. Furthermore if the repeat of the differential ∂ , i.e. ∂^2 always turns to null, $\partial^2 = 0$ (other equivalent conditions are $\bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$), then the almost complex manifold becomes complex manifold. (The proof of these statements is omitted in this paper.) In this paper only complex manifolds are concerned.

$\Gamma_\alpha^\beta = \Gamma_{\gamma\alpha}^\beta dz^\gamma$ is called $[1, 0]$ type connection, and $\bar{\Gamma}_{\bar{\alpha}}^{\bar{\beta}} = \bar{\Gamma}_{\bar{\gamma}\bar{\alpha}}^{\bar{\beta}} d\bar{z}^{\bar{\gamma}}$ is called $[0, 1]$ type connection. The group transformation cannot change the type of connection. In this paper, for simplicity we employ only $[1, 0]$ type connections, and $[0, 1]$ type connections are thus all trivial.

B. The complex manifold specified by the general linear group $GL(n, \mathcal{C})$

It seems that the expression in section IV is only developed for real geometry, but in fact it also pertains to complex geometry. Here we iterate the results in terms of complex geometry.

First of all, let's assume that we have chosen a metric for the complex space,

$$A(\bar{\psi}, \psi) = A_{\bar{\alpha}\beta} d\bar{\psi}^\alpha d\psi^\beta . \quad (5.5)$$

It is an additional requirement to real geometry that the indices α and β belong to two different types, one is normal component, the other is its complex conjugate, $\alpha \rightarrow \bar{\alpha}$. The reason why we choose this particular metric form will be explained in the next section. Following the process of the last section, let's make the connection preserve the metric $A(\bar{\psi}, \psi)$. Having chosen the $\frac{\partial}{\partial \bar{\psi}^\alpha}, \frac{\partial}{\partial \psi^\beta}$ as basis $\{e_{\bar{\alpha}}, e_\beta\}$, the result is easily obtained from Eq.(4.19) that

$$dA_{\alpha\bar{\beta}} - A_{\gamma\bar{\beta}} \Gamma_\alpha^\gamma - A_{\alpha\bar{\gamma}} \Gamma_{\bar{\beta}}^{\bar{\gamma}} = 0 . \quad (5.6)$$

In obtaining the above equation, we have used the following equations due to the parallel displacement

$$d\xi^{\bar{\alpha}} + \Gamma_{\bar{\gamma}}^{\bar{\alpha}} \xi^{\bar{\gamma}} = 0 , \quad (5.7a)$$

$$d\eta^\beta + \Gamma_\gamma^\beta \eta^\gamma = 0 , \quad (5.7b)$$

where $\xi^{\bar{\alpha}}$ and η^β are components for the vector X and Y , $X = \xi^{\bar{\alpha}} e_{\bar{\alpha}}$, $Y = \eta^\beta e_\beta$. In analogy with the form (4.22), Eq.(5.6) can be expressed in the following matrix form

$$dA = \Gamma^t \cdot A + A \cdot \bar{\Gamma} \quad (5.8)$$

where $\bar{\Gamma}$ refers to the matrix with components $\Gamma_{\bar{\alpha}}^{\bar{\gamma}}$. Performing exterior product on the above equation turns the left hand trivial, and

$$\begin{aligned} 0 &= ddA = d\Gamma^t \cdot A - \Gamma^t \wedge dA + dA \cdot \bar{\Gamma} + A \cdot d\bar{\Gamma} \\ &= d\Gamma^t \cdot A - \Gamma^t \wedge (\Gamma^t \cdot A) - \Gamma^t \wedge (A \cdot \bar{\Gamma}) \\ &\quad + (\Gamma^t \cdot A + A \cdot \bar{\Gamma}) \wedge \bar{\Gamma} + A \cdot d\bar{\Gamma} \\ &= \Omega^t \cdot A + A \cdot \bar{\Omega}, \end{aligned} \quad (5.9)$$

where $\Omega^t = d\Gamma^t - \Gamma^t \wedge \Gamma^t$, in which the minus sign comes from the transpose of matrices. Eq. (5.9) suggests the curvature matrix and metric matrix are of anticommute. In Eq.(5.8), if Γ^t is the $[1, 0]$ type, then $\bar{\Gamma}$ is the $[0, 1]$ type. Bearing in mind that $d = \partial + \bar{\partial}$, Eq.(5.8) gives rise to

$$\partial A = \Gamma^t \cdot A , \quad (5.10a)$$

i.e.

$$\Gamma = (A^t)^{-1} \cdot \partial A^t . \quad (5.10b)$$

The inverse of matrix A is so defined that

$$A_{\alpha\bar{\beta}} A^{\bar{\beta}\gamma} = \delta_\alpha^\gamma , \quad A^{\bar{\alpha}\gamma} A_{\gamma\bar{\beta}} = \delta_{\bar{\beta}}^{\bar{\alpha}} . \quad (5.11)$$

Then due to (5.10b) the component form of Γ is

$$\Gamma_\alpha^\beta = A^{\beta\bar{\gamma}} \frac{\partial A_{\bar{\gamma}\alpha}}{\partial z^i} dz^i . \quad (5.12)$$

Consequently the curvature form (4.23) and (4.24) certainly hold, with the variables now being complex

$$\Omega = (\Omega_\alpha^\beta) = d\Gamma - \Gamma \wedge \Gamma , \quad (4.23)$$

from $\Gamma_\alpha^\beta = \Gamma_{k\alpha}^\beta dz^k$, and taking into account the definition of the components form of curvature, $\Omega_\alpha^\beta = R_{\alpha jk}^\beta d\bar{z}^j \wedge dz^k$, it is straightforward to obtain

$$R_{\alpha jk}^\beta = \Gamma_{k\alpha, j}^\beta - \Gamma_{j\alpha, k}^\beta + \Gamma_{j\gamma}^\beta \Gamma_{k\alpha}^\gamma - \Gamma_{k\gamma}^\beta \Gamma_{j\alpha}^\gamma . \quad (4.24)$$

Here the meaning of $d\bar{z}^j$ is the same as that of $d\bar{\psi}^j$, and likewise for dz^k . From this form it is obvious that the curvature is antisymmetric with respect to the last two indices j, k . Except the requirement Eq. (5.4) due to analytic condition, the manifold in virtue of $GL(n, \mathcal{C})$ adds no more constraint to the connection.

C. The complex manifold specified by the unitary group $U(n, \mathcal{C})$

On the basis of the above general complex manifold, if we demand the metric matrix A satisfy

$$(A^t)^* = A \quad (5.13)$$

with components form as $A_{\bar{\alpha}\beta} = \bar{A}_{\alpha\bar{\beta}}$ i.e. if A -matrix is Hermitian conjugate to itself, then the metric is called the Hermitian metric. All the components of the Hermitian metric form the Hermitian matrix (Hermitian operator). This kind of operator is well known in quantum mechanics and has many good properties. To keep the Hermitian metric $d\bar{\psi}^\alpha A_{\bar{\alpha}\beta} d\psi^\beta$ and corresponding curvature invariant, the transformation group must be a Unitary group $U(n, \mathcal{C})$.

D. The Kähler manifold

Additionally, as for an Hermitian manifold, if the connection $\Gamma_{\alpha\gamma}^\beta$ is symmetric with respect to the indices α and γ , the manifold then is called Kähler manifold. The transformation group for Kähler manifold is $SU(n, \mathcal{C})$.

E. Independent components of curvature and Ricci tensor

To summarize the above results relevant to curvature, we note that it subjects to three constraints as the following.

- (I) As shown in Eq. (5.4), the form of connection like $\Gamma_{\alpha\bar{\beta}}^\gamma$ and $\Gamma_{\alpha\beta}^{\bar{\gamma}}$ is trivial,
- (II) We only use $[1, 0]$ type connection, so the $[0, 1]$ type connection $\Gamma_{\bar{\alpha}\beta}^\gamma = \Gamma_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} = 0$,
- (III) The curvature form is $[1, 1]$ type, so the forms $R_{\delta\alpha\beta}^\gamma$ or $R_{\delta\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}$ vanish.

Under these constraints, one can calculate the following curvature components one by one according to the explicit form (4.24): $R_{\bar{\beta}jk}^\alpha$, $R_{\bar{\beta}\bar{j}k}^\alpha$, $R_{\beta\bar{j}k}^\alpha$, $R_{\beta j\bar{k}}^\alpha$, $R_{\bar{\beta}j\bar{k}}^\alpha$, $R_{\beta j\bar{k}}^\alpha$, $R_{\bar{\beta}jk}^{\bar{\alpha}}$, $R_{\bar{\beta}\bar{j}k}^{\bar{\alpha}}$, $R_{\beta j\bar{k}}^{\bar{\alpha}}$, $R_{\beta\bar{j}k}^{\bar{\alpha}}$. Finally one concludes that only four of them are nonzero

$$R_{\bar{\beta}jk}^\alpha = \Gamma_{k\bar{\beta},\bar{j}}^\alpha, \quad (5.14a)$$

$$R_{\beta j\bar{k}}^\alpha = -\Gamma_{j\beta,\bar{k}}^\alpha, \quad (5.14b)$$

$$R_{\bar{\beta}jk}^{\bar{\alpha}} = \Gamma_{k\bar{\beta},\bar{j}}^{\bar{\alpha}}, \quad (5.14c)$$

$$R_{\beta j\bar{k}}^{\bar{\alpha}} = -\Gamma_{j\beta,\bar{k}}^{\bar{\alpha}}. \quad (5.14d)$$

The above four equations hold independent of the torsion [The torsion is defined as $T_{\gamma\beta}^\alpha = \Gamma_{\gamma\beta}^\alpha - \Gamma_{\beta\gamma}^\alpha$] since the evaluating process has nothing to do with it. When the torsion is absent, the first two curvatures in Eq. (5.14) display more symmetries $R_{\bar{\beta}jk}^\alpha = R_{k\bar{j}\beta}^\alpha$, $R_{\beta j\bar{k}}^\alpha = R_{j\beta\bar{k}}^\alpha$.

The constraints certainly affect the Ricci tensor. Based on Eq.(5.14), the Ricci tensors are obtained by tracing Ω in Eq. (4.23),

$$\begin{aligned} R_{\bar{\beta}jk}^\alpha &\xrightarrow{\alpha=\beta} R_{\bar{j}k}, R_{\beta j\bar{k}}^\alpha \xrightarrow{\alpha=\beta} R_{j\bar{k}}, \\ R_{\bar{\beta}jk}^{\bar{\alpha}} &\xrightarrow{\bar{\alpha}=\bar{\beta}} R_{\bar{j}k}, R_{\beta j\bar{k}}^{\bar{\alpha}} \xrightarrow{\bar{\alpha}=\bar{\beta}} R_{j\bar{k}}. \end{aligned} \quad (5.15)$$

The equation suggests only two independent forms of Ricci tensor, $R_{\bar{j}k}$ and $R_{j\bar{k}}$, exist. Furthermore, the curvature component $R_{\bar{\beta}jk}^\alpha$ is antisymmetric with respect to the indices \bar{j} and k , i.e. $R_{\bar{j}k} = -R_{k\bar{j}}$, whence only one form of Ricci tensor would occur. We will return to this antisymmetry in section IX.

Now the Ricci tensor defined by (5.15) can be explicitly written as

$$R_{\bar{\alpha}\beta} = R_{\gamma\bar{\alpha}\beta}^\gamma = \Gamma_{\beta\gamma,\bar{\alpha}}^\gamma \text{ (or } \Gamma_{\beta\bar{\gamma},\bar{\alpha}}^{\bar{\gamma}}) = \frac{\partial \Gamma_{\beta\gamma}^\gamma}{\partial \bar{z}^\alpha} = \frac{\partial^2 \ln A}{\partial \bar{z}^\alpha \partial z^\beta}, \quad (5.16)$$

here the notation A stands for determinant $\det(A_{\bar{\alpha}\beta})$, and the last step is obtained by applying the differential relation of a matrix determinant

$$A_{,\nu} = A A^{\beta\bar{\alpha}} A_{\bar{\alpha}\beta,\nu} \quad (5.17)$$

to the derivatives of $\Gamma_{\beta\gamma}^\gamma$

$$\frac{\partial \Gamma_{\beta\gamma}^\gamma}{\partial \bar{z}^\alpha} = \frac{\partial}{\partial \bar{z}^\alpha} (A^{\gamma\bar{\rho}} \frac{\partial A_{\bar{\rho}\gamma}}{\partial z^\beta}) = \frac{\partial}{\partial \bar{z}^\alpha} (\frac{\partial}{\partial z^\beta} \ln A) . \quad (5.18)$$

The above discussions set no limit on transformation group, hence pertain to general situations, say, $GL(n, \mathcal{C})$. For the Hermitian or Kähler manifold, the independent components of curvature and Ricci tensor should decrease. If $A_{\bar{\alpha}\beta}$ is Hermitian, then $A_{\bar{\alpha}\beta}^* = A_{\beta\bar{\alpha}}$ and $\det(A_{\bar{\alpha}\beta}) = \det(A_{\bar{\alpha}\beta}^\dagger)$, applying (5.14a), (5.14d) or (5.14b), (5.14c) to (5.18), the following expression can be achieved

$$R_{\bar{\alpha}\beta}^* = R_{\alpha\bar{\beta}} . \quad (5.19)$$

The specification of Kähler manifold adds no more constraint to the form of Ricci tensor except the symmetric indices of connection in equation (5.14a), (5.14b).

Now let's count the number of the independent components of the Ricci tensor. Apart from the fact that the two indices are antisymmetric, the group $GL(4, \mathcal{C})$ adds no constraints. So in this case the set $\{\bar{\alpha}, \beta\}$, with α, β being over 1, 2, 3, 4, has totally 16 elements. While $\{\alpha, \beta\}$ need not be taken into account for antisymmetry. The freedom of $A_{\bar{\alpha}\beta}$, however, is totally 32 components, so it requires 16 gauge conditions to solve the field Eq. (9.8). Furthermore, if the manifold is Hermitian, i.e. the Eq.(5.19) holds, $R_{\bar{\alpha}\beta}^* = -R_{\beta\bar{\alpha}}$, then the elements of set $\{\bar{\alpha}, \beta\}$ decrease to 6, and the freedom of $A_{\bar{\alpha}\beta}$ also shrinks to 20, hence 14 additional equations are needed to resolve the $A_{\bar{\alpha}\beta}$, almost the same as that of $GL(4, \mathcal{C})$.

VI. THE METRIC FOR FERMION FIELD

A quantum fermion field is customarily expressed by Dirac spinor $\psi(x)$, and the inner product of $\psi(x)$ is prescribed as

$$\bar{\psi}(x)\psi(x) = \psi^\dagger(x)\gamma_0\psi(x) , \quad (6.1)$$

where

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \quad (6.2)$$

To be consistent with conventional quantum field theory (CFQT), here the $\bar{\psi}(x)$ is defined as $\psi^\dagger(x)\gamma_0$, which is different from previous denotation $\bar{\psi}_\alpha(x) = \psi_\alpha^*(x)$. Henceforth we distinguish the difference by whether or not the subscripts or superscripts are used: if used, then the latter definition is available, otherwise the former definition holds. The former definition $\bar{\psi}(x) = \psi^\dagger(x)\gamma_0$ is meaningful in making the inner product $\bar{\psi}(x)\psi(x)$ invariant under the transformation group $SL(2, \mathcal{C})$, which is the spinor-representation of Lorentz group. The invariance is a direct corollary of Lorentz invariance of Dirac equation [2].

To apply the differential geometry, let's generalize the inner product. To do so the definition of metric should be valid only within a very small region of the complex space, e.g. the inner product $\bar{\psi}(x)\psi(x)$ should be generalized to $d\bar{\psi}(x)d\psi(x)$, and only in this sense does the inner product remain invariant under the transformation of group $SL(2, \mathcal{C})$. Any infinitesimal transformation is now performed on $d\psi(x)$ instead of on $\psi(x)$. $d\psi(x)$ can be viewed as plane wave locally and $\psi(x)$ has conventional meaning on a larger scale.

The above inner product holds when there is no interaction, if any interaction arises, the product has to be interpreted by a general metric form $A_{\alpha\bar{\beta}}d\psi^\alpha d\bar{\psi}^{\bar{\beta}}$. The above form $d\bar{\psi}(x)d\psi(x)$ is only a special case while $A_{\alpha\bar{\beta}} = \eta_{\alpha\bar{\beta}}$, $\eta = 1$ for $\alpha = 1, 2$ and $\eta = -1$ for $\alpha = 3, 4$. In conclusion, the metric of the quantum field is demanded to be

$$A(\bar{\psi}, \psi) = A_{\alpha\bar{\beta}}d\psi^\alpha d\bar{\psi}^{\bar{\beta}} = d\psi^\alpha A_{\alpha\bar{\beta}}d\bar{\psi}^{\bar{\beta}} , \quad (6.3)$$

which does not violate experiences from QED.

VII. TWO TYPES OF CURVING IN QUANTUM FIELDS

In a two-dimension (2-D) plane, the curves of hyperbolic or elliptic type can be interpreted by the equations like $a^2 x^2 - b^2 y^2 = 1$ or $a^2 x^2 + b^2 y^2 = 1$. General linear transformations in 2-D [Known to be $GL(2, R)$ group. They have the general form $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$, with c_{ij} being real numbers] can't change the types of curves, since the performance of linear transformations has two identical manners: one is to change the objective curves; the other is to change the coordinate axes. Applying the latter manner, the types of the curves are obviously reserved. In other words, under general linear transformations within 2-D, if a curve in 2-D is ever a type, it will forever be the type. In this paper we take into account only the quadratically homogeneous forms for variables, so considering these two types is enough.

The above argument is expected to hold also in four dimension space (4-D). It becomes complicate in this case, since a 4-D curve being hyperbolic in one projected plane may be an ellipse in another. So the definition of hyperbolic or elliptic type has to be extended: For a 4-D curve, if there exists at least one 2-D plane so that the curve projected into it is hyperbolic, then curve is called hyperbolic, otherwise it is elliptic. In our concerns the 4-D interval [In this paper "metric" means the same as "interval"] $c^2 t^2 - x^2 - y^2 - z^2$ in special relativity is hyperbolic type, because if the equation $c^2 t^2 - x^2 - y^2 - z^2 = \text{constant}$ is projected into any 2-D subspace including the time axis, the resultant curve is a hyperbolic type. Lorentz group preserves the interval $c^2 t^2 - x^2 - y^2 - z^2$, so the hyperbolic characteristic is also the main feature of Lorentz group, which is called noncompact in terminology of group theory. In contrast, the group $SO(4)$ is compact for it preserve the interval $c^2 t^2 + x^2 + y^2 + z^2$, which is a elliptic type. Instead of requiring the transformation groups to preserve the interval, now we only require the types of interval to be preserved. Then we can find that hyperbolic characteristic of the interval $c^2 t^2 - x^2 - y^2 - z^2$ will not change under the general linear group $GL(4, R)$, similar to the above 2-D case. Generally, to keep the type of an n -D interval of quadratic homogeneous form, that the transformation is linear is sufficient. Here and hereafter the explanations of concepts relevant to Group are rough and only for later application.

The above definitions of hyperbolic and elliptic type in 2-D are for the global space. In a local region of the whole space, the two types should be interpreted by $a^2 dx^2 - b^2 dy^2 = 1$ and $a^2 dx^2 + b^2 dy^2 = 1$ respectively. Correspondingly, if Special Relativity is treated locally, then the metric is

$$c^2 dt^2 - dx^2 - dy^2 - dz^2. \quad (7.1)$$

In a small local region, the infinitesimal transformations of $GL(4, R)$ cannot change the types of the curves. Therefore, even though a transformation changes (7.1) to a general form $g^{\alpha\beta} dx_\alpha dx_\beta$, by which we cannot tell its type directly, the type of the metric $g^{\alpha\beta} dx_\alpha dx_\beta$ remains the same as Eq. (7.1).

Now we extend the above discussions and knowledge to complex space. In following three paragraphs, we construct the **parallelisms** between the metrics of real space and complex space, as well as the **parallelisms** between the groups preserving them, then generalize our understanding of hyperbolic and elliptic type in real space to the understanding of complex space.

Let's consider the complex-space metrics defined in the last section, $\psi^\dagger(x) \gamma_0 \psi(x) = u_1^* u_1 + u_2^* u_2 - u_3^* u_3 - u_4^* u_4$, where $\psi(x)$ is a complex spinor $\psi^\dagger = (u_1^*, u_2^*, u_3^*, u_4^*)$. This metric is invariant under the transformations of group $SL(2, \mathcal{C})$, \mathcal{C} means that the elemental variables in the group are complex ones. [3, 4]. It can be verified [3] that there exists a two-to-one mapping between all the elements of the group $SL(2, \mathcal{C})$ and all the elements of the proper, orthochronous Lorentz group. Hereby the group $SL(2, \mathcal{C})$ is viewed as **complex parallelism** of the Lorentz group (which is born from physics in real space-time).

Now let's elucidate the relationship between group $SL(2, \mathcal{C})$ and aforementioned metric $\psi^\dagger(x) \gamma_0 \psi(x)$. The quantities which are transformed according to lowest-dimensional nontrivial representation of $SL(2, \mathcal{C})$ are called two-component *spinors*, which are doublets as states for spin. On the basis of higher dimensional representation of $SL(2, \mathcal{C})$ other spinors with more components can be constructed. In our case the four-component spinor $\psi(x)$ is transformed according to named $D^{(\frac{1}{2}, \frac{1}{2})}$ representation of $SL(2, \mathcal{C})$ [involving γ -matrices], their relationship can be derived from the covariance of Dirac equation and we omit all the relevant details here. Also it can be confirmed that the metric form $\psi^\dagger(x) \gamma_0 \psi(x)$ remains unaltered under the transformation of $D^{(\frac{1}{2}, \frac{1}{2})}$ representation of $SL(2, \mathcal{C})$. In a word, the metric $\psi^\dagger(x) \gamma_0 \psi(x)$ is the **complex parallelism** to the real interval $c^2 t^2 - x^2 - y^2 - z^2$. Consequently the **complex parallelism** to Eq. (7.1) should be

$$d\bar{\psi}(x) d\psi(x). \quad (7.2)$$

Now we know $\psi^\dagger(x) \gamma_0 \psi(x)$ —and hence $d\bar{\psi}(x) d\psi(x)$ —is hyperbolic. However, we don't know what is the largest group—which includes $SL(2, \mathcal{C})$ as a subgroup—that makes the type of metric $\psi^\dagger(x) \gamma_0 \psi(x)$ unchanged. Searching for what is exactly the largest group would be a tedious work and might simultaneously deviate us from the main line of

developing this theory. In this paper we simply assume the latest group is $GL(4, \mathcal{C})$ (in fact must be a subgroup of $GL(4, \mathcal{C})$) and bearing in mind that it is for hyperbolic type.

The **metric parallelism** between elliptic types of real space and complex space can be constructed by directly generalizing each of real axes to a complex one. We are familiar with the metric defined in the 3-D space $x^2 + y^2 + z^2$, which is invariant under the rotation group $SO(3)$. Suppose a complex space spanned by wave functions with only three components, $(\psi_1(x), \psi_2(x), \psi_3(x))$, corresponding to metric $x^2 + y^2 + z^2$ and consistent with the probability in quantum mechanics, we conclude that its metric should be of the form $\psi_1^*(x)\psi_1(x) + \psi_2^*(x)\psi_2(x) + \psi_3^*(x)\psi_3(x)$, where the integration over the configuration space is implied. To preserve the probability in the whole complex space, the quantum mechanics requires any transformations performed on the wave function (ψ_1, ψ_2, ψ_3) should be elements of special unitary group $SU(3)$ or its subgroups. $SU(n)$ groups are compact, just as $SO(n)$. Hereby the metric $\psi_1^*(x)\psi_1(x) + \psi_2^*(x)\psi_2(x) + \psi_3^*(x)\psi_3(x)$ and group $SU(n)$ are viewed as the **complex parallelism** to that of elliptic type of real space. The caution should be practiced that it is not perfect to generalize the rotation transformation in real space directly to conservation of angular momentum as done in quantum mechanics to achieve the same result that metric $\psi_1^*(x)\psi_1(x) + \psi_2^*(x)\psi_2(x) + \psi_3^*(x)\psi_3(x)$ and $SU(n)$ are elliptic type. Since the Lorentz group includes a spatial rotation part besides the boost part, from the Noether theorem of quantum field theory one can also derive the conservation of angular momentum. Then there will be confusions between the two types. The degrees of freedom of elliptic type presented in this paper are those degrees of freedom such as electro-charges, isospins, or colours etc., other than spins and spatial angular momentum. The largest group to preserve the elliptic type of the metric $\sum_i \psi_i^*(x)\psi_i(x)$ is certainly $U(n)$ groups.

We have endowed the Lorentz symmetry and special Unitary symmetry with hyperbolic type and elliptic type, the corresponding types of metric defined according to these two symmetries remain stable even after the groups are appropriately extended. We know $U(n)$ group will not change the type of metric $\sum_i \psi_i^*(x)\psi_i(x)$, and a subgroup of $GL(4, \mathcal{C})$ will not change the type of $\psi^\dagger(x)\gamma_0\psi(x)$. It is the extended groups that provide more degrees of freedom over which the complex space spanned by wave functions is curved. Corresponding to the two symmetries, there are two types of curvings. Since for a fermion, such as quark, it has both the curving characteristics—hyperbolic type and elliptic type—spinor and colour, we should combine the two types of metric into one general form

$$A_{\alpha\beta}^{ac} d\bar{\psi}_a^\alpha d\psi_c^\beta. \quad (7.3)$$

where we generalize $\psi^\alpha(x)$ to $\psi_c^\alpha(x)$, $c = 1, 2, 3$ are colour indices, α, β -are spinor indices. The metric matrix $(A_{\alpha\beta}^{ac})$ may be written separately as a hyperbolic part multiplied by an elliptic part.

VIII. MOTION EQUATION FOR FERMIONS

A. General motion equation for fermions

After reviewing the geometrical method, let's turn to its application to physical problems.

As an equal level observer, assumedly an electron can test another electron without losing any information. To an observer what is the motion equation of the observed electron? For an observer in space-time the right equation is certainly the Schrödinger equation or the Dirac equation. But as an equal level observer, it constructs the equation as follows: there always exists a complex local frame for the observer in which the observed electron looks like a plane wave (a free electron), in spite of the existence of interaction—similar to that happens in General Relativity—saying that there always exists a local frame in space-time in which the observed particle looks like a free one moving along a straight line, in spite of the gravitation. In terms of geometry, it means that the motion of electron is just a geodetic line in complex space, i.e., the parallel transport (displacement).

Conventionally a free electron is described by plane wave $u(\vec{p}) e^{ip \cdot x}$, $u(\vec{p})$ is Dirac spinor. The complex space here should be of 4-dimension since $u(\vec{p})$ possesses four components, furthermore, it should be hyperbolic as argued in the preceding section. In this respect the transformation group for the space should be $GL(4, \mathcal{C})$. Since the complex space is continuously spanned by components of local spinor $\{\psi^1, \psi^2, \psi^3, \psi^4\}$, its natural bases are $\{e_\alpha = \frac{\partial}{\partial \psi^\alpha}, e_{\bar{\alpha}} = \frac{\partial}{\partial \bar{\psi}^\alpha}\}$. Consequently, $X = \xi^{\bar{\alpha}} e_{\bar{\alpha}} = d\bar{\psi}^\alpha \frac{\partial}{\partial \bar{\psi}^\alpha}$, then the parallel displacement (5.6a) $(d\xi^{\bar{\alpha}} + \Gamma_{\bar{\gamma}}^{\bar{\alpha}} \xi^{\bar{\gamma}}) = 0$ turns out to be

$$d\bar{\psi}^\alpha + \Gamma_{\bar{\gamma}}^{\bar{\alpha}} d\bar{\psi}^{\bar{\gamma}} = 0. \quad (8.1)$$

Similarly, another equation for ψ^α holds,

$$(d\psi^\beta + \Gamma_{\bar{\gamma}}^\beta d\bar{\psi}^{\bar{\gamma}}) = 0. \quad (8.2)$$

Above two equations are the general forms of motion equation for fermions.

B. The rule of calculating the differential d

In Eq. (8.2) the component $d\psi^\beta$ is an infinitesimal change of ψ^β relative to its neighboring points, which is determined by ψ^β and the transformation performed on it. Suppose that after a transformation the wave function ψ^β changes to ψ'^β , then resembling the form of Eq.(3.7), the wave function ψ'^β can be expressed by variables ψ^β ,

$$\psi'^\beta = \psi'^\beta(\psi^1, \bar{\psi}^1; \psi^2, \bar{\psi}^2; \dots, \psi^n, \bar{\psi}^n) . \quad (8.3)$$

Now the differential form $d\psi^\beta$ can be written explicitly as

$$d\psi'^\beta = \frac{\partial \psi'^\beta}{\partial \psi^\gamma} d\psi^\gamma + \frac{\partial \psi'^\beta}{\partial \bar{\psi}^\gamma} d\bar{\psi}^\gamma , \quad (8.4)$$

The derivative $\frac{\partial \psi'^\beta}{\partial \psi^\gamma}$ in the above equation however, is not operable in practical calculation.

Practically, we want to know how far the Eqs. (8.1), (8.2) deviate from the conventional Dirac equation. Or to put it alternatively, how can the terms in Dirac equation be finally restored in local region by reducing aforementioned parallel displacement? To obtain a form of Dirac equation, let's first recall the replacement used in GR: $d^2x^\mu + \Gamma^\mu_{\sigma\rho} dx^\sigma dx^\rho = 0 \rightarrow \frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{dt} \frac{dx^\sigma}{dt} = 0$. That reminds us to replace the differential operator d in Eqs. (8.1), (8.2) by some forms of derivatives with respect to space-time. In view of the quadratic form of Eqs. (8.1), (8.2), it is helpful to know the following quadratic form [2] of Dirac equation before we do some replacement of operator d ,

$$[(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) - \frac{i}{2}e\gamma_\mu\gamma_\nu F^{\mu\nu}]\psi = m^2\psi , \quad (8.5)$$

where $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field tensor of common sense. In what follows, we put $e = 1$. Then the explicit form of Eq. (8.5) is

$$\partial_\mu \partial^\mu \psi + i \partial_\mu A^\mu \psi + i A_\mu \partial^\mu \psi - A_\mu A^\mu \psi + \frac{i}{2} \gamma_\mu \gamma_\nu F^{\mu\nu} \psi = -m^2 \psi , \quad (8.6a)$$

By using a weaker Lorentz condition $\partial_\mu A^\mu \psi = 0$ of Gupta and Bleuler [5] instead of the original one $\partial_\mu A^\mu = 0$, the above equation changes to

$$\square \psi + i A_\mu \partial^\mu \psi - A_\mu A^\mu \psi + \frac{i}{2} \gamma_\mu \gamma_\nu F^{\mu\nu} \psi = -m^2 \psi . \quad (8.6b)$$

As a reasonable approximation to leading-order QED, the quadratic terms of A^μ , such as $A_\mu A^\mu \psi$ in the above equation and similar terms in our following calculation will be temporarily omitted.

Now let's turn to treating the equation (8.2). First let's substitute the explicit form of $\Gamma^\alpha_{\beta\gamma}$, $A^{\alpha\bar{\rho}} \frac{\partial A_{\bar{\rho}\gamma}}{\partial z^\beta}$, into the equation (8.2), concerning only the second term, it yields

$$\begin{aligned} \Gamma^\alpha_{\beta\gamma} d\psi^\beta d\psi^\gamma &= A^{\alpha\bar{\rho}} \frac{\partial A_{\bar{\rho}\gamma}}{\partial \psi^\beta} d\psi^\beta d\psi^\gamma \\ &= A^{\alpha\bar{\rho}} dA_{\bar{\rho}\gamma} d\psi^\gamma . \end{aligned} \quad (8.7)$$

Then the form of motion equation (8.2) becomes

$$dd\psi^\alpha + A^{\alpha\bar{\rho}} dA_{\bar{\rho}\gamma} d\psi^\gamma = 0. \quad (8.8)$$

Extending the discussion of Eqs. (6.1)~(6.3) straightforwardly by considering the vertex $\gamma^\mu A_\mu$ used in QED, we can interpret the metric tensor $A_{\bar{\rho}\gamma}$ with interaction as $(A_{\bar{\rho}\gamma})_{4 \times 4} = \gamma_0 + \gamma_0 \gamma^\mu A_\mu$ (Henceforth we continue to use A^μ to express the field potential. Take care not to confuse it with the metric tensor $A^{\alpha\beta}$). It is easy to verify that $(A^{\alpha\bar{\rho}})_{4 \times 4} = \gamma_0 - \gamma^\mu \gamma_0 A_\mu$. To obtain a form of Dirac equation, comparing the second term of Eq. (8.8) with the term $\gamma_\mu \gamma_\nu F^{\mu\nu}$ in Eq. (8.6), we find it is effective to replace d by $\gamma_\mu \partial^\mu$. Obviously, the replacement $d \rightarrow \gamma_\mu \partial^\mu$ performed to the first term of Eq. (8.8) directly induces $dd \rightarrow \square$.

Furthermore, by analyzing the dimensions of Eq. (8.8), we find the dimensions of the first term and that of the second term are not equal. In the natural units, the first term of the above equation has the dimension of energy square if assuming ψ dimensionless. As for the second term, the form $A^{\alpha\bar{\rho}} = \gamma_0 - \gamma^\mu \gamma_0 A_\mu$ can be simplified as $1 + B$, and $dA_{\bar{\rho}\gamma} [\gamma_\mu \partial^\mu (\gamma_\nu A^\nu)]$ is energy square too, but the derivative performed on ψ also contribute the dimension of an energy, so an extra energy dimension exists. To remedy the unequal dimensions of two sides and according to our

experience of treating the Schrödinger equation with a nonlocal interaction-potential [6], we add a line integral with respect to space-time $x^\mu = (t, \vec{x})$ to the second term. The integration is accompanied by the purely imaginary number i according to requirement restoring the Dirac equation. Now the form of equation (8.8) becomes

$$dd\psi^\alpha + i \int_{x^\mu(-\infty)}^{x^\mu(t)} A^{\alpha\bar{\rho}} dA_{\bar{\rho}\gamma} d\psi^\gamma dx = 0, \quad (8.9)$$

Consistently, the infinitesimal integral measurement dx should also be linked with γ -Matrix, assumed as $dx \rightarrow \gamma_\lambda dx^\lambda$.

C. Restoring the terms in Dirac equation

Now let's evaluate the results of the second term $\int_{x^\mu(-\infty)}^{x^\mu(t)} A^{\alpha\bar{\rho}} dA_{\bar{\rho}\gamma} d\psi^\gamma dx$ in Eq. (8.9). It is convenient to write out the integral in terms of matrix

$$\begin{aligned} & \int (1 - \gamma_\mu A^\mu) \gamma_\sigma \partial^\sigma (1 + \gamma_\nu A^\nu) \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda \\ &= \int \gamma_\sigma \partial^\sigma (\gamma_\nu A^\nu) \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda \\ & \quad - \int \gamma_\mu A^\mu \gamma_\sigma \partial^\sigma (\gamma_\nu A^\nu) \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda, \end{aligned} \quad (8.10)$$

here $\tilde{\psi} = \{\psi^1, \psi^2, \psi^3, \psi^4\}$ is a four-component vector, $A^{\alpha\bar{\rho}}$ and $A_{\bar{\rho}\gamma}$ all in their matrix forms, $1 - \gamma_\mu A^\mu$ and $1 + \gamma_\nu A^\nu$. We will mainly deal with the first integral in (8.10), since the second term includes two A^μ factors we temporarily omit it as second-order perturbation. The calculation involves the formulae of γ -matrices in simplifying the product $\gamma_\sigma \gamma_\nu \gamma_\rho \gamma_\lambda$. The detailed evaluation on the first integral is put to Appendix A, here we only show the result in (A. 7)

$$\begin{aligned} & \int \gamma_\sigma \partial^\sigma (\gamma_\nu A^\nu) \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda \\ &= A_\nu \partial^\nu \psi + \frac{1}{2} \gamma_\mu \gamma_\nu \tilde{F}^{\mu\nu} \psi + \frac{1}{2} \gamma_\mu \gamma_\nu F^{\mu\nu} \psi \\ & \quad + \frac{1}{2} \gamma_\rho \gamma_\lambda \int (\partial_\nu A^\nu) (dx^\lambda \partial^\rho - dx^\rho \partial^\lambda) \psi \\ & \quad + i \epsilon_{\sigma\rho\lambda\mu} \gamma_5 \gamma^\mu \int \partial^\sigma (\gamma_\nu A^\nu) \partial^\rho \psi dx^\lambda - \int A_\nu \partial_\lambda \partial^\nu \psi dx^\lambda - \frac{1}{2} \gamma_\mu \gamma_\nu \int \tilde{F}^{\mu\nu} \partial_\lambda \psi dx^\lambda \\ & \quad - \frac{1}{2} \gamma_\mu \gamma_\nu \int \partial_\lambda F^{\mu\nu} \psi dx^\lambda - \gamma_\mu \gamma_\nu \int \partial^\sigma A^\mu \partial_\sigma \psi dx^\nu \end{aligned} \quad (8.11)$$

where $\tilde{F}^{\mu\nu} = A^\mu \partial^\nu - A^\nu \partial^\mu$. Submitting the integral result back into (8.9) yields

$$\begin{aligned} & \square \psi + i A_\mu \partial^\mu \psi + \frac{i}{2} \gamma_\mu \gamma_\nu F^{\mu\nu} \psi + \frac{i}{2} \gamma_\mu \gamma_\nu \tilde{F}^{\mu\nu} \psi \\ & + \frac{i}{2} \gamma_\rho \gamma_\lambda \int (\partial_\nu A^\nu) (dx^\lambda \partial^\rho - dx^\rho \partial^\lambda) \psi \\ & - \epsilon_{\sigma\rho\lambda\mu} \gamma_5 \gamma^\mu \int \partial^\sigma (\gamma_\nu A^\nu) \partial^\rho \psi dx^\lambda - i \int A_\nu \partial_\lambda \partial^\nu \psi dx^\lambda - \frac{i}{2} \gamma_\mu \gamma_\nu \int \tilde{F}^{\mu\nu} \partial_\lambda \psi dx^\lambda \\ & - \frac{i}{2} \gamma_\mu \gamma_\nu \int \partial_\lambda F^{\mu\nu} \psi dx^\lambda - i \gamma_\mu \gamma_\nu \int \partial^\sigma A^\mu \partial_\sigma \psi dx^\nu \\ & = 0, \end{aligned} \quad (8.12)$$

it is found that the terms $i A_\nu \partial^\nu \psi$ and $\frac{i}{2} \gamma_\mu \gamma_\nu F^{\mu\nu} \psi$ are just those required by (8.6b). However, quite a few other terms like $\frac{i}{2} \gamma_\mu \gamma_\nu \tilde{F}^{\mu\nu} \psi$, $-i \int A_\nu \partial_\lambda \partial^\nu \psi dx^\lambda$, etc. accompany the required ones. Some of the redundant terms, such as $\frac{i}{2} \gamma_\rho \gamma_\lambda \int (\partial_\nu A^\nu) (dx^\lambda \partial^\rho - dx^\rho \partial^\lambda) \psi$, $\epsilon_{\sigma\rho\lambda\mu} \gamma_5 \gamma^\mu \int \partial^\sigma (\gamma_\nu A^\nu) \partial^\rho \psi dx^\lambda$, $i \gamma_\mu \gamma_\nu \int \partial^\sigma A^\mu \partial_\sigma \psi dx^\nu$, with integrand relevant to local angular momentum $(dx^\lambda \partial^\rho - dx^\rho \partial^\lambda)$, which is assumed second-order small in perturbation theory according to our numerical experience, can be omitted temporarily. The contributions of remainder terms like $\frac{i}{2} \gamma_\mu \gamma_\nu \tilde{F}^{\mu\nu} \psi$,

$i \int A_\nu \partial_\lambda \partial^\nu \psi dx^\lambda$, $\frac{i}{2} \gamma_\mu \gamma_\nu \int \tilde{F}^{\mu\nu} \partial_\lambda \psi dx^\lambda$, $\frac{i}{2} \gamma_\mu \gamma_\nu \int F^{\mu\nu} \partial_\lambda \psi dx^\lambda$ cannot be judged reasonably, so these terms cannot be thrown off and may show their significance in some situations when the wavelength of fermions is sufficiently long. For instance, $\tilde{F}^{\mu\nu}$ is an operator, and it renders the results of $\frac{i}{2} \gamma_\mu \gamma_\nu \tilde{F}^{\mu\nu} \psi$ dependent on the variation of the wave function ψ . Accordingly this sort of terms is possibly relevant to the nonlocal effect of Aharonov-Bohm type [7]. The terms $\frac{i}{2} \gamma_\mu \gamma_\nu \int \tilde{F}^{\mu\nu} \partial_\lambda \psi dx^\lambda$, $\frac{i}{2} \gamma_\mu \gamma_\nu \int F^{\mu\nu} \partial_\lambda \psi dx^\lambda$ would behave in similar ways. But the term $i \int A_\nu \partial_\lambda \partial^\nu \psi dx^\lambda$, without the factor $\gamma_\mu \gamma_\nu$ before it, is the exceptional case as discussed in the following.

Comparing the Eq. (8.12) with Eq. (8.6), it is noted that there lacks a mass term $-m^2 \psi$ in the right hand of Eq. (8.12). There are two ways to remedy this flaw. First, one can directly add a mass term to the right hand of equation (8.9), then the equation (8.2) would have a nonzero term in its right hand too. That obviously violates our original hypothesis that motion equation is just the geodetic line. Another way out is to accept the term $i \int A_\nu \partial_\lambda \partial^\nu \psi dx^\lambda$ as the mass term. The reason is that according to the Klein-Gordon equation of the next section the double derivatives on potential A_ν would induce a mass factor and the integration $\int A_\nu dx^\lambda$ would induce a phase factor depending on closed paths (loops) of fermions [7], thus the total effect of the term $\int A_\nu \partial_\lambda \partial^\nu \psi dx^\lambda$ seems equal to a mass term for motion equation of fermions. If the paths are not closed, then the mass of fermions would become dependent on potential A_ν . The claims of this paragraph are only qualitative, further confirmation is necessary. The relevant works are in process.

Apart from the above qualitative display, all of the four nontrivial terms, $\frac{i}{2} \gamma_\mu \gamma_\nu \tilde{F}^{\mu\nu} \psi$, $i \int A_\nu \partial_\lambda \partial^\nu \psi dx^\lambda$, $\frac{i}{2} \gamma_\mu \gamma_\nu \int \tilde{F}^{\mu\nu} \partial_\lambda \psi dx^\lambda$ and $\frac{i}{2} \gamma_\mu \gamma_\nu \int F^{\mu\nu} \partial_\lambda \psi dx^\lambda$, are surely nonlocal since they obviously ruin the local conservation law $\partial_\mu j^\mu = 0$ [where $j^\mu = \psi \gamma^\mu \psi$], which can be directly derived from the conventional Dirac equation (8.5).

D. Brief summary for this section

If the approximation $(A_{\bar{\rho}\gamma})_{4 \times 4} = \gamma_0 + \gamma_0 \gamma^\mu A_\mu$ is not used here, the replacement $d \rightarrow \gamma_\mu \partial^\mu$ would not be reasonable. In general case the matrix $(A_{\bar{\rho}\gamma})_{4 \times 4}$ should be expanded to all possibilities of γ -matrices (the total number of matrices is 16) without respecting only the vector form of interaction. But we think there leaves little arbitrariness to extend the theory to include more sorts of interaction. If surely the approximation $(A_{\bar{\rho}\gamma})_{4 \times 4} = \gamma_0 + \gamma_0 \gamma^\mu A_\mu$ holds for fundamental interactions, then accordingly we would extend the replacement $d \rightarrow \gamma_\mu \partial^\mu$ to $d \rightarrow \lambda^a \gamma_\mu \partial_a^\mu$, λ^a -Gellmann matrices, when we are concerned about the colour interaction between quarks. That will cause more integrations and other intricate terms to enter into Eq. (8.12). We will not detail that issue here.

In conventional quantum field theory (CQFT), we only regard the part $F^{\mu\nu}$ being responsible for the physical process. However, in view of the nonlocal (curving) effects, it has been stated in Ref. [8] that the $F^{\mu\nu}$ is not complete in describing all physics. Even under our rough approximation, it can be seen in Eq. (8.12) that more terms are present than required. It includes not only the normal terms $F^{\mu\nu}$ but also some other terms on account of nonlocal effect.

Moreover, the definition of the connection and thus the physical meaning of A^μ now are not the same as those in CQFT, since in CQFT, A^μ (potential) is assumed as connection and $F^{\mu\nu}$ as the curvature tensor. But here the A^μ is viewed only as components of metric tensor and $F^{\mu\nu}$ as connection (force). Correspondingly the dimension of motion equation here is also different from that of Schrödinger equation, the former is a force, and the latter a potential. A discrepancy of energy dimension appears under natural unit.

IX. FIELD EQUATION FOR BOSONS

A. The rule of calculating the derivative $\frac{\partial}{\partial \bar{z}^j} \frac{\partial}{\partial z^k}$

The differential problem in fact has arisen in calculating the terms $dd\psi^\beta$ and $\Gamma_{a\gamma}^\beta d\psi^\alpha d\bar{\psi}^\gamma$. As a hypothesis, we have required the wave function to change with respect to space-time variables x_μ , with which the terms in Dirac equation for electrodynamics are restored. However, in fact, since we are discussing the property in complex space, the differential $dd\psi^\beta$ should change with respect to complex variables as showed in Eq. (8.3), i.e. $dd\psi^\beta = \frac{\partial^2 \psi^\beta}{\partial \bar{\psi}^\alpha \partial \bar{\psi}^\gamma} d\bar{\psi}^\alpha \wedge d\bar{\psi}^\gamma$. Making every coefficient of $d\psi^\rho \wedge d\bar{\psi}^\sigma$ vanish, then Eq.(8.2) yields

$$\frac{\partial^2 \psi'^\beta}{\partial \bar{\psi}^\rho \partial \bar{\psi}^\sigma} + \Gamma_{a\gamma}^\beta \frac{\partial \psi'^\alpha}{\partial \bar{\psi}^\rho} \frac{\partial \bar{\psi}'^\gamma}{\partial \bar{\psi}^\sigma} = 0 \quad (9.1)$$

This equation should be thoroughly respected by the electron as an equal level observer to another observed electron.

Although it is impossible for the electron to release all information it carries by projecting its complex space to our space-time, $\psi^\alpha \rightarrow x^\mu$, we have to do so to get some physical observables in space-time. Just for the physical reason

and for consistence, in this section we directly make the replacements of the differential forms dd and $dAdB$, or the equivalent forms $\frac{\partial^2}{\partial\psi^\rho\partial\psi^\sigma}$ and $\frac{\partial A}{\partial\psi^\rho}\frac{\partial B}{\partial\psi^\sigma}$, as follows:

$$dd \text{ or } \frac{\partial^2}{\partial\psi^\rho\partial\psi^\sigma} \longrightarrow \square \quad (9.2a)$$

$$dAdB \text{ or } \frac{\partial A}{\partial\psi^\rho}\frac{\partial B}{\partial\psi^\sigma} \longrightarrow \frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial x_\mu} \quad (9.2b)$$

B. The antisymmetry of Ricci tensor and the operator \square

This subsection is devoted to explaining a subtle but important aspect in calculating the curvature tensor. It can be seen from the definition of Eq.(4.24) that the curvature tensor $R_{\alpha jk}^\beta$ is explicitly antisymmetric with respect to the indices j and k . However, the antisymmetry seems lost from the expression (5.14). Now let us impose that antisymmetry on the interpretation of Eq.(5.14) and see what will be the result. Substitute the explicit form of connection Γ_{β}^α into Eq.(5.14), the components of curvature become explicit functions with respect to metric tensor:

$$\begin{aligned} R_{\beta jk}^\alpha &= \frac{\partial}{\partial \bar{z}^j}(A^{\alpha\bar{\gamma}}\frac{\partial A_{\bar{\gamma}\beta}}{\partial z^k}), \quad R_{\beta j\bar{k}}^\alpha = -\frac{\partial}{\partial \bar{z}^k}(A^{\alpha\bar{\gamma}}\frac{\partial A_{\bar{\gamma}\beta}}{\partial \bar{z}^j}), \\ R_{\beta jk}^{\bar{\alpha}} &= \frac{\partial}{\partial \bar{z}^j}(A^{\bar{\alpha}\gamma}\frac{\partial A_{\gamma\bar{\beta}}}{\partial z^k}), \quad R_{\beta j\bar{k}}^{\bar{\alpha}} = -\frac{\partial}{\partial \bar{z}^k}(A^{\bar{\alpha}\gamma}\frac{\partial A_{\gamma\bar{\beta}}}{\partial \bar{z}^j}). \end{aligned} \quad (9.3)$$

We only calculate the first one as an example, the others can be obtained by properly changing the indices. Impose the antisymmetric property on the first equation of Eq.(9.3)

$$\frac{\partial}{\partial \bar{z}^j}(A^{\alpha\bar{\gamma}}\frac{\partial A_{\bar{\gamma}\beta}}{\partial z^k}) = -\frac{\partial}{\partial z^k}(A^{\alpha\bar{\gamma}}\frac{\partial A_{\bar{\gamma}\beta}}{\partial \bar{z}^j}), \quad (9.4)$$

then the left hand and the right hand can be changed respectively to

$$left = -\frac{\partial}{\partial \bar{z}^j}[\frac{\partial A^{\alpha\bar{\gamma}}}{\partial z^k}A_{\bar{\gamma}\beta}] = -\frac{\partial A^{\alpha\bar{\gamma}}}{\partial z^k}\frac{\partial A_{\bar{\gamma}\beta}}{\partial \bar{z}^j} - A_{\bar{\gamma}\beta}\frac{\partial}{\partial \bar{z}^j}\frac{\partial}{\partial z^k}A^{\alpha\bar{\gamma}}, \quad (9.5a)$$

$$right = -\frac{\partial A^{\alpha\bar{\gamma}}}{\partial z^k}\frac{\partial A_{\bar{\gamma}\beta}}{\partial \bar{z}^j} - A^{\alpha\bar{\gamma}}\frac{\partial}{\partial z^k}\frac{\partial}{\partial \bar{z}^j}A_{\bar{\gamma}\beta}, \quad (9.5b)$$

the equality of the two sides directly gives

$$A^{\alpha\bar{\gamma}}\frac{\partial}{\partial z^k}\frac{\partial}{\partial \bar{z}^j}A_{\bar{\gamma}\beta} = A_{\bar{\gamma}\beta}\frac{\partial}{\partial \bar{z}^j}\frac{\partial}{\partial z^k}A^{\alpha\bar{\gamma}}, \quad (9.6a)$$

i.e.

$$A^{\alpha\bar{\gamma}}\partial\bar{\partial}A_{\bar{\gamma}\beta} = A_{\bar{\gamma}\beta}\bar{\partial}\partial A^{\alpha\bar{\gamma}}. \quad (9.6b)$$

The relation seems trivial in this form, but making the replacement of (9.2) results in

$$A^{\alpha\bar{\gamma}}\square A_{\bar{\gamma}\beta} = A_{\bar{\gamma}\beta}\square A^{\alpha\bar{\gamma}}, \quad (9.7)$$

which is a reasonable result telling that the operator \square is Hermitian. So now we can insist on the antisymmetry in Ricci tensor without worrying about any unexpected contradiction.

As a byproduct of applying the above argument to Ricci tensor (5.18), a rule is gained that for any product of two derivatives like $\frac{\partial}{\partial \bar{z}^j}\frac{\partial}{\partial \bar{z}^k}A$, A arbitrary, the result should be antisymmetric if permuting $\frac{\partial}{\partial \bar{z}^j}$ and $\frac{\partial}{\partial \bar{z}^k}$, $\frac{\partial}{\partial \bar{z}^j}\frac{\partial}{\partial \bar{z}^k}A = -\frac{\partial}{\partial \bar{z}^k}\frac{\partial}{\partial \bar{z}^j}A$.

C. Field Equation

Now let's introduce the field equation for bosons, e.g. photons in Electrodynamics. For a boson field without any source, we are concerned about the case when the interaction is absent, i.e., the connection (force) of complex space is trivial, and thus all the components of curvature tensor $R_{\beta j k}^\alpha$ vanish. In that case the boson field satisfies the equation $R_{\beta j k}^\alpha = 0$. The equation will not change under certain transformations in question (e. g. $GL(4, C)$ group for photons) since $R_{\beta j k}^\alpha$ is a tensor. That is to say, the field equation will not change with the appearance of interaction. But as a general equation for a boson field the condition $R_{\beta j k}^\alpha = 0$ seems too strict, so we introduce a weaker constraint by merely demanding that Ricci tensors vanish,

$$R_{\alpha\beta} = 0 . \quad (9.8a)$$

That not only holds for free fields, but also is expected to hold while a source appears at a point infinitely far away. As in GR, we can add the source term to the right hand of equation, whose physical meaning will be clarified later,

$$R_{\alpha\beta} = \kappa S_{\alpha\beta} . \quad (9.8b)$$

The coefficient κ can be determined by comparing it with the Klein-Gordon equation. Next we will study what can be derived from this field equation, and what are its differences with the Klein-Gordon equation as well.

D. The Approximation of Metric Matrix With the Scalar Part Only

We had better resolve the field equation (9.8) precisely before discussing the field property. But here we expect to understand the field properties by substituting some good approximations into the field equation.

As argued in section VI, in the absence of interaction the metric matrix is of the form

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \quad (9.9)$$

and after a period of interaction, according to the perturbative theory (in this whole paper we respect the perturbative results of CQFT), the matrix evolves into

$$\gamma_0 + \gamma_0 \gamma^\mu A_\mu = \begin{pmatrix} 1 + A_0 & -\vec{\sigma} \cdot \vec{A} \\ -\vec{\sigma} \cdot \vec{A} & -1 + A_0 \end{pmatrix} . \quad (9.10)$$

If only the electronic part is present, Eq. (9.10) gives

$$A_{\alpha\beta} \simeq \begin{pmatrix} 1 + A_0 & 0 & 0 & 0 \\ 0 & 1 + A_0 & 0 & 0 \\ 0 & 0 & -1 + A_0 & 0 \\ 0 & 0 & 0 & -1 + A_0 \end{pmatrix} . \quad (9.11)$$

Moreover, if we choose the large components approximation, which is validated by QED, then the form of $A_{\alpha\beta}$ yields

$$A_{\alpha\beta} \simeq \begin{pmatrix} 1 + A_0 & 0 & 0 & 0 \\ 0 & 1 + A_0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \quad (9.12)$$

Now we briefly review the reasonability of the above approximation. The large components and the small components appear in solution of Dirac equation when both the electric part (scalar potential) and magnetic part (vector part) of the boson field are small and the kinetic energy of fermion is very low: In general, the Dirac equation can be written [2],

$$H\psi = (m + W)\psi = \vec{\alpha} \cdot (\vec{P} - \vec{A})\psi + \beta m\psi + e\phi\psi , \quad (9.13)$$

where W is kinetic energy, ϕ the scalar potential and \vec{A} the vector potential. Divide the four components ψ into two parts composed respectively of the first two components and the last two components,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \quad (9.14)$$

with this denotation the Dirac equation is readily reduced into two equations,

$$\begin{aligned} (m + W)\psi_a &= \vec{\sigma} \cdot (\vec{P} - \vec{A})\psi_b + m\psi_a + e\phi\psi_a \\ (m + W)\psi_b &= \vec{\sigma} \cdot (\vec{P} - \vec{A})\psi_a - m\psi_b + e\phi\psi_b. \end{aligned} \quad (9.15)$$

After carrying out ψ_b as $\psi_b = (2m + W - \phi)^{-1} \vec{\sigma} \cdot (\vec{P} - \vec{A})\psi_a$, we see that when W , ϕ and \vec{A} are all small, the two components ψ_a and ψ_b obey the relation

$$\psi_b \sim \frac{\vec{P}}{m}\psi_a \sim \frac{v}{c}\psi_a. \quad (9.16)$$

So in the large-component-approximation, the potential A_0 in the last two diagonal elements in (9.11) can be ignored reasonably since it contributes to $\psi^\alpha A_{\alpha\bar{\beta}} \bar{\psi}^\beta$ the terms two orders less than that of the first two diagonal elements. Consequently the approximation (9.12) does hold.

Under this approximation, the Ricci tensor reads

$$R_{\alpha\beta} = \frac{\partial^2 \ln \det \begin{pmatrix} 1 + A_0 & 0 & 0 & 0 \\ 0 & 1 + A_0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}{\partial \psi^\alpha \partial \psi^\beta}. \quad (9.17)$$

Replacing the differential $\frac{\partial^2}{\partial \psi^\alpha \partial \psi^\beta}$ following Eq. (9.2), and assuming that the scalar field does not vary with time, then up to $(A^0)^3$ order the field equation $R_{\alpha\beta} = 0$ turns out to be

$$\vec{\nabla}^2 A^0 = 0 \quad (9.18)$$

which is just the Poisson equation satisfied by the electric field in vacuum. In the next subsection it will be shown that the approximation without the electric field is also heuristic.

E. The Approximation of Metric Matrix With the Vector Part Only and a Conjecture on the Origin of Mass for Bosons

Now let's turn to the approximation without electric field or source term, i.e., the case that only radiation field exists. Computing the similar form of Ricci tensor in (9.17) by using the metric form (9.10) and making $A_0 = 0$, we have

$$\square \vec{A}^2 = 0. \quad (9.19)$$

We will treat the equation explicitly to construct its relation to the electric and magnetic energy forms \vec{E}^2 and \vec{B}^2 , where $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. The calculation of (9.19) includes two parts

$$\frac{\partial^2}{\partial t^2} \vec{A}^2 = 2\left(\frac{\partial}{\partial t} \vec{A}\right)^2 + 2\vec{A} \cdot \frac{\partial^2}{\partial t^2} \vec{A}, \quad (9.20)$$

and

$$\begin{aligned} \vec{\nabla}^2 \vec{A}^2 &= \vec{\nabla} \cdot (\vec{\nabla} \vec{A}^2) = \vec{\nabla} \cdot (2\vec{A} \cdot \vec{\nabla} \vec{A}) \\ &= 2(\vec{\nabla} \vec{A}) : (\vec{\nabla} \vec{A}) + 2\vec{A} \cdot \vec{\nabla}^2 \vec{A}, \end{aligned} \quad (9.21)$$

where the underlines denote the inner products between the vectors. And the sign $:$ is the tensor product for dyads, for example, $\vec{p} \cdot \vec{\sigma}_1 \vec{k} \cdot \vec{\sigma}_2 = \underline{\vec{p}\vec{k}} : \vec{\sigma}_1 \vec{\sigma}_2$. If an inner product is emphasized by underlines then the regular product order between dyads is not followed, e.g. $\underline{\vec{p}\vec{k}} : \underline{\vec{\sigma}_1 \vec{\sigma}_2} = \vec{k} \cdot \vec{\sigma}_1 \vec{p} \cdot \vec{\sigma}_2$. Usually we only use the underlines while the gradient operator $\vec{\nabla}$ appears. Now let's express the magnetic energy \vec{B}^2 in terms of vector potential

$$\begin{aligned} \vec{B}^2 &= (\vec{\nabla} \times \vec{A})^2 = (\vec{\nabla} \times \vec{A}) \cdot (\vec{\nabla} \times \vec{A}) \\ &= \vec{\nabla} \cdot (\vec{A} \times (\vec{\nabla} \times \vec{A})) + \vec{A} \cdot (\vec{\nabla} \times (\vec{\nabla} \times \vec{A})) \end{aligned} \quad (9.22)$$

To further simplify the Eq. (9.22), the following relations are useful,

$$\begin{aligned} 2(\vec{A} \times (\vec{\nabla} \times \vec{A})) &= \vec{\nabla}(\vec{A} \cdot \vec{A}) - 2(\vec{A} \cdot \vec{\nabla})\vec{A} \\ &= 2\underline{\vec{A}} \cdot (\vec{\nabla} \underline{\vec{A}}) - 2(\vec{A} \cdot \vec{\nabla})\vec{A} \end{aligned} \quad (9.23a)$$

$$\begin{aligned} \vec{\nabla} \cdot (\vec{A} \times (\vec{\nabla} \times \vec{A})) &= \vec{\nabla} \cdot [\underline{\vec{A}} \cdot (\vec{\nabla} \underline{\vec{A}}) - (\vec{A} \cdot \vec{\nabla})\vec{A}] \\ &= (\vec{\nabla} \underline{\vec{A}}) : (\vec{\nabla} \underline{\vec{A}}) + \vec{A} \cdot \vec{\nabla}^2 \vec{A} - \vec{\nabla} \underline{\vec{A}} : \underline{\vec{\nabla}} \vec{A} - (\vec{A} \cdot \vec{\nabla})(\vec{\nabla} \cdot \vec{A}) \end{aligned} \quad (9.23b)$$

$$\vec{A} \cdot \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{A} \cdot [\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}] = (\vec{A} \cdot \vec{\nabla})(\vec{\nabla} \cdot \vec{A}) - \vec{A} \cdot \vec{\nabla}^2 \vec{A} \quad (9.23c)$$

Applying relations in (9.23) to (9.22), we obtain

$$\vec{B}^2 = (\vec{\nabla} \underline{\vec{A}}) : (\vec{\nabla} \underline{\vec{A}}) - \vec{\nabla} \underline{\vec{A}} : \underline{\vec{\nabla}} \vec{A} = (\vec{\nabla} \underline{\vec{A}}) : (\vec{\nabla} \underline{\vec{A}}) \quad (9.24)$$

The last step holds for transverse fields in the Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = \underline{\vec{A}} \cdot \underline{\vec{\nabla}} \vec{A} = 0$. Now the Eq. (9.20) and Eq. (9.21) can be simplified to

$$\frac{\partial^2}{\partial t^2} \vec{A}^2 = 2\vec{E}^2 + 2\vec{A} \cdot \frac{\partial^2}{\partial t^2} \vec{A} \quad (9.25a)$$

$$-\vec{\nabla}^2 \vec{A}^2 = -2\vec{B}^2 - 2\vec{A} \cdot \vec{\nabla}^2 \vec{A} \quad (9.25b)$$

Combine them into (9.19)

$$\square \vec{A}^2 = 2(\vec{E}^2 - \vec{B}^2) + 2\vec{A} \cdot \square \vec{A} = 0 \quad (9.26)$$

Consequently, the following equation holds,

$$\vec{A} \cdot \square \vec{A} = \vec{B}^2 - \vec{E}^2 \stackrel{d}{=} -m^2 \vec{A}^2 \quad (9.27)$$

Heuristically, the Eq. (9.27) suggests if the inequality

$$\vec{E}^2 - \vec{B}^2 \neq 0 \quad (9.28)$$

holds, then the equation for field \vec{A} can automatically gain a mass term

$$\vec{A} \cdot (\square + m^2) \vec{A} = 0 \quad (9.29)$$

where the Klein-Gordon equation takes shape. For Electrodynamics, since $\vec{E}^2 - \vec{B}^2$ coincidentally and accurately vanishes, the net mass of photon is zero. In QED, the Lagrangian density for photon field is

$$L = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \quad (9.30)$$

the field tensor $F_{\mu\nu}(x)$ is defined in the preceding section. In terms of \vec{E} and \vec{B} , the Lagrangian can be transformed to

$$L = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) = -\frac{1}{2} (\vec{E}^2 - \vec{B}^2) = -\frac{1}{2} m^2 \vec{A}^2 \quad (9.31)$$

Coincident with Lagrange undetermined multiplier method, now it is natural to extend the Lagrangian of QED by adding a mass term

$$L = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \frac{1}{2}m^2 A_\mu A^\mu, \quad (9.30)$$

from which the Klein-Gordon equation for massive bosons can be obtained by using variational method. In the frame of this paper we don't respect gauge invariance of the Lagrangian. But the variational method would always be valid from mathematical viewpoint.

Now let's clarify the meaning of source term in Eq. (9.8b). In quantum field theory the Klein-Gordon equation with source term is $\square \vec{A} = \vec{j}$, hence according to Eq. (9.29)

$$\vec{A} \cdot (\square + m^2) \vec{A} = \vec{A} \cdot \vec{j} = \vec{A} \cdot \bar{\psi} \vec{\gamma} \psi, \quad (9.31)$$

the right hand is just the interaction Hamiltonian. Remembering that we have in Eq. (9.2) replaced the derivatives with respect to ψ by the derivatives with respect to space-time, and noticing the form $\frac{\partial}{\partial \psi} = \frac{\partial}{\partial x} \frac{\partial x}{\partial \psi}$, the above equation and Eq. (9.17) suggest that the source term corresponding to Eq. (9.8) should be of a double integrals with $\not{A} = A_\mu \gamma^\mu$ as integrand and $(\gamma_\nu dx^\nu)^2$ as its infinitesimal integral measurement. We define $\tilde{A}_{\alpha\beta} = \int \int A_\mu \gamma^\mu (\gamma_\nu dx^\nu)^2$, then

$$R_{\alpha\beta} = \lambda \tilde{A}_{\alpha\beta}, \quad (9.32)$$

here the $\tilde{A}_{\alpha\beta}$ refers to a source field induced by fermions, though it looks formally like from A_μ . The term $\vec{E}^2 - \vec{B}^2$ should not be associated with $\tilde{A}_{\alpha\beta}$ since it is derived from the equation for free boson field.

F. The general forms of field equation in space-time

In above subsections the discussions are applicable in fact only to $U(1)$ case, in which A_μ has no intrinsic degrees of freedom except spatial polarization. As stated in the previous section, if other intrinsic degrees are involved [e. g. see (7.3), $A_\mu \rightarrow A_{\alpha\beta}^{ac}$], the replacement $d \rightarrow \gamma_\mu \partial^\mu$ will not hold any longer. Consequently, the replacement of Eq. (9.2) becomes abated. Instead we should employ the replacement $d \rightarrow \tau^a \gamma_\mu \partial_a^\mu$, τ^a being generators of a Unitary group. When the group is $SU(2)$, whose representation can use Pauli matrices, the replacement performing on double differential dd will induce the similar form to $U(1)$ case, $dd \rightarrow \square_1^2 + \square_2^2 + \square_3^2$. In this case the field equation in space-time would be

$$(\square_1^2 + \square_2^2 + \square_3^2) \det(A_{\alpha\beta}^{ac}) = 0. \quad (9.33)$$

However, in $SU(3)$ case, the Gellmann matrices $\{\lambda^a, a = 1, \dots, 8\}$ have not the property of $(\lambda^a k_a)(\lambda^b k_b) = k_1^2 + k_2^2 + \dots + k_8^2$, so that the cross-terms don't vanish. Therefore, in $SU(3)$ case, the field equation in space-time can written only as

$$(\lambda^a \gamma_\mu \partial_a^\mu)(\lambda^b \gamma_\nu \partial_b^\nu) \det(A_{\alpha\beta}^{ac}) = \lambda^a \lambda^b \gamma_\mu \gamma_\nu \partial_a^\mu \partial_b^\nu \det(A_{\alpha\beta}^{ac}) = 0. \quad (9.34)$$

X. THE BIANCHI IDENTITY AND REMARKS ON CONSERVATION LAW

Make exterior differential to both sides of Eq. (4.23), one gets the following Bianchi identity

$$d\Omega = \Omega \wedge \Gamma - \Gamma \wedge \Omega, \quad (10.1)$$

and its matrix form is

$$d\Omega_k^l = \Omega_h^l \wedge \Gamma_k^h - \Gamma_h^l \wedge \Omega_k^h. \quad (10.2)$$

Write it out in components form

$$\begin{aligned} \frac{\partial R_{bjk}^a}{\partial z^m} dz^m \wedge dz^j \wedge dz^k &= R_{cij}^a dz^i \wedge dz^j \wedge \Gamma_{mb}^c dz^m - \Gamma_{mc}^a dz^m \wedge R_{bij}^c dz^i \wedge dz^j \\ &= (R_{cmi}^a \Gamma_{jb}^c - \Gamma_{mc}^a R_{bij}^c) dz^m \wedge dz^i \wedge dz^j. \end{aligned} \quad (10.3)$$

On the other hand, the covariant differential of R_{bij}^a can be obtained by definition

$$R_{bij;m}^a = \frac{\partial R_{bij}^a}{\partial z^m} + \Gamma_{mc}^a R_{bij}^c - \Gamma_{mb}^c R_{cij}^a - \Gamma_{mi}^c R_{bcj}^a - \Gamma_{mj}^c R_{bic}^a . \quad (10.4)$$

Combining the above two, one gets

$$R_{bij;m}^a dz^m \wedge dz^i \wedge dz^j = -(\Gamma_{mi}^c R_{bcj}^a + \Gamma_{mj}^c R_{bic}^a) dz^m \wedge dz^i \wedge dz^j . \quad (10.5)$$

In the case that torsion is absent, the right side is equal to zero, hence

$$R_{b[ij;m]}^a = 0 . \quad (10.6)$$

The bracket $[ij; m]$ means the left side also includes the terms with the indices of cyclic permutation. Substituting the nonzero components in Eq.(5.14) one by one to the above equation, for instance substituting the (5.14a), it becomes

$$\begin{aligned} R_{\beta[\bar{j}k;m]}^\alpha &= R_{\beta\bar{j}k;m}^\alpha + R_{\beta m\bar{j};k}^\alpha + R_{\beta km;\bar{j}}^\alpha \\ &= R_{\beta\bar{j}k;m}^\alpha + R_{\beta m\bar{j};k}^\alpha \stackrel{\text{contracting } \alpha \text{ and } \beta}{=} R_{\bar{j}k;m} + R_{m\bar{j};k} = 0 . \end{aligned} \quad (10.7)$$

The other three forms of Eq.(5.14) will give rise to the same form of the Ricci identity $R_{\bar{j}k;m} + R_{m\bar{j};k} = 0$. It has been proved in Sec.V that the metric is invariant under covariant differential, $A_{\alpha\beta;j} = D_j A_{\alpha\beta} = 0$. So with the definition of scalar curvature $R = A^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}$, the Ricci identity possesses the following form

$$(A^{m\bar{j}} R_{m\bar{j}})_{;k} + (A^{m\bar{j}} R_{\bar{j}k})_{;m} = 0 , \quad (10.8)$$

i.e.,

$$R_{;k} - R_{k;m}^m = 0 . \quad (10.9)$$

Applying the relation $A_{\alpha\beta;j} = 0$ to above equation, with $A^{\bar{k}} R_k^m = R^{\bar{m}}$, the Eq. (10.9) can be changed to

$$(R^{\bar{m}} - A^{\bar{m}} R)_{;m} = 0 . \quad (10.10)$$

Now the field equation can be rewritten by regarding the Bianchi identity,

$$R^{\bar{m}} - A^{\bar{m}} R = 0 , \quad (10.11)$$

its equivalence to Eq. (9.8) can be demonstrated as follows: contracting the indices of Eq. (10.11) results in $R - 4R = 0$, then substituting $R = 0$ back to Eq. (10.11), finally the Eq. (9.8) is restored. Furthermore, the above equation can be extended to include source term

$$R^{\bar{m}} - A^{\bar{m}} R = \kappa S^{\bar{m}} . \quad (10.12)$$

As mentioned in the last section, the approximation (9.10) is good for perturbative theories. In such case, the metric reads [9]

$$A_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} , \quad (10.13)$$

where the $h_{\alpha\beta}$ is required to be small comparing with $\eta_{\alpha\beta}$, whose nonvanishing elements are only those diagonal ones, being 1, 1, -1, -1. Here the linear metric $\eta_{\alpha\beta}$ is used to raise or lower the indices of metric and curvature of linear part, e.g. $\eta^{\bar{\alpha}\beta} h_{\beta\bar{\gamma}} = h_{\bar{\gamma}}^{\bar{\alpha}}$. Only in this sense, the following defined $R_{\bar{\alpha}\beta}^L$ is different from the normal definition of the Ricci tensor $R_{\bar{\alpha}\beta}$, though they have almost the same form. The linear part of the Ricci tensor is

$$R_{\bar{\alpha}\beta}^L = \frac{\partial^2 h_{\bar{\alpha}}^\lambda}{\partial \bar{z}^\alpha \partial z^\beta} . \quad (10.14)$$

The rest part then is

$$R_{\bar{\alpha}\beta} - R_{\bar{\alpha}\beta}^L = \frac{\partial}{\partial \bar{z}^\alpha} (h^{\bar{\delta}\gamma} \frac{\partial h_{\gamma\bar{\delta}}}{\partial z^\beta}) . \quad (10.15)$$

Similar to the treatment in General Relativity [7], we separate the left hand of the field Eq. (10.12) into two parts $R_{\bar{\rho}m} - A_{\bar{\rho}m}R = R_{\bar{\rho}m}^{\mathbf{L}} - \eta_{\bar{\rho}m}R^{\mathbf{L}} - \kappa\Pi_{\bar{\rho}m}$, then Eq. (10.12) can be written

$$R_{\bar{\rho}m}^{\mathbf{L}} - \eta_{\bar{\rho}m}R^{\mathbf{L}} = \kappa(\Pi_{\bar{\rho}m} + S_{\bar{\rho}m}) . \quad (10.16)$$

We can prove that the differential performed on the left hand of Eq. (10.16) gives zero

$$\frac{\partial}{\partial z^m}(R_{\bar{\rho}m}^{\mathbf{L}} - \eta_{\bar{\rho}m}R^{\mathbf{L}}) = 0 . \quad (10.17)$$

The first term is zero since the antisymmetric tensor $\frac{\partial}{\partial z^m}R^{\mu\bar{\nu}}$ (Indices $m, \bar{\nu}$ are antisymmetric) is multiplied by symmetric tensor $\eta_{m\bar{\nu}}$, i.e., $R_{\bar{\rho}m}^{\mathbf{L}} = \eta_{\bar{\rho}\mu}\eta_{m\bar{\nu}}R^{\mu\bar{\nu}}$; and the second term is trivial since the symmetric tensors $\eta_{n\bar{\nu}}, \eta_{\bar{\sigma}\mu}$ and $\eta^{\bar{\sigma}n}$ are accompanied with the antisymmetric tensor $R^{\mu\bar{\nu}}$, so $\eta^{\bar{\sigma}n}\eta_{\bar{\sigma}\mu}\eta_{n\bar{\nu}}R^{\mu\bar{\nu}} = R^{\mathbf{L}}$ is directly zero before differential is performed.

From (10.10), (10.12) we conclude that the source term $S_{\bar{\rho}m}$ is conservative in covariant sense, $S_{\bar{\rho}m;m} = 0$. Whereas from (10.16), (10.17) we conclude that the sum $(\Pi_{\bar{\rho}m} + S_{\bar{\rho}m})$ is conservative in sense of common differential, $\frac{\partial}{\partial z^m}(\Pi_{\bar{\rho}m} + S_{\bar{\rho}m}) = 0$. The two sorts of conservation directly bring out three conclusions:

(I) As for the existence of the conserved equation

$$\frac{\partial}{\partial z^m}(\Pi_{\bar{\rho}m} + S_{\bar{\rho}m}) = 0 \quad (10.18)$$

it is natural to view the sum $\Pi_{\bar{\rho}m} + S_{\bar{\rho}m}$ as the total energy-momentum tensor including both the fermion part $S_{\bar{\rho}m}$ and the boson part $\Pi_{\bar{\rho}m}$, which suggests in a complex-infinitesimal-local-region the energy and momentum may transfer between $S_{\bar{\rho}m}$ and $\Pi_{\bar{\rho}m}$.

(II) Since the $R_{\bar{\rho}m}^{\mathbf{L}}$ will not be a tensor for the loss of nonlinear terms, it can be inferred from (10.16) that the boson part $\Pi_{\bar{\rho}m}$ does not satisfy $\Pi_{\bar{\rho}m;m} = 0$ (this equation and (10.18) can't be satisfied at the same time), and thus can't be a locally conserved quantity. Partly for this reason the theory in this paper is **nonlocal**.

(III) It seems that the aforementioned mass terms $\vec{E}^2 - \vec{B}^2$ should be included in the boson self-energy term $\Pi_{\bar{\rho}m}$, but the last word on that requires more investigation. If that is the case, what about the remaining parts in $\Pi_{\bar{\rho}m}$?

The conclusions of this section are obtained in the absence of torsion. If torsion is present, then the forms of formulae will be more complicated.

XI. PHYSICAL REGION

A. The Curved Space for Electromagnetic Field

From now on we will work in the approximation of metric tensor shown in (9.10):

$$(A_{\alpha\beta})_{4\times 4} = \gamma_0 + \gamma_0\gamma^\mu A_\mu = \begin{pmatrix} 1 + A_0 & -\vec{\sigma} \cdot \vec{A} \\ -\vec{\sigma} \cdot \vec{A} & -1 + A_0 \end{pmatrix}, \quad (9.10)$$

which is justified by the success of CQFT for perturbative interaction. The metric, as stated in section VI, is

$$A(X, Y) = d\psi^\alpha A_{\alpha\bar{\beta}} d\bar{\psi}^{\bar{\beta}}, \quad (11.1)$$

correspondingly the Ricci curvature for it is

$$R_{\bar{\alpha}\beta} = \frac{\partial^2 \ln A}{\partial \bar{z}^\alpha \partial z^\beta}. \quad (11.2)$$

The present goal is to determine the physical region on the basis of the above formulae. Substituting Eq. (9.10) to the Eq. (11.2), we find the main quantity to be evaluated is the determinant of matrix $(A_{\alpha\bar{\beta}})_{4\times 4}$. The determinant of (9.10) is

$$|A_{\bar{\alpha}\beta}| = 1 - A_0^2 + \vec{A}^2. \quad (11.3)$$

Suggested by the horizon of black hole in GR, the horizon-like boundary of the physical region may exist also for those fields with the same theoretical frame as GR. As a tentative step, let's first examine the physical region (or

alternatively, the singularity) for electrons (whose dynamics is governed by QED) to get some rules and then apply them to quarks (The dynamics is mostly governed by QCD).

From the form of Eq.(11.2), it is noted that if

$$|A_{\bar{\alpha}\beta}| = 1 - A_0^2 + \vec{A}^2 = 0, \quad (11.4)$$

then the logarithm function in Ricci tensor becomes divergent. The solutions of the equation are defined as the singularities of Ricci tensor. In the large-component-approximation and $\vec{A} \sim 0$, we obtain

$$A_0 \sim 1. \quad (11.5)$$

For the electron in a hydrogen-atom, A_0 is the Coulomb form $\frac{1}{r}$, then Eq. (11.5) leads to $r \sim 1$. In the atomic (natural) unit, that means the length of r is the average radius of the ground state of the hydrogen atom, which suggests that the zero singularity of logarithm function possibly symbolizes one side of physical region. On the other hand, if $r \rightarrow \infty$, then $1 - \frac{1}{r^2} \rightarrow 1$, the electron tends to be free (asymptotically free). In summary we achieve the following conclusion held for electrons,

$$|A_{\bar{\alpha}\beta}| = 1 - A_0^2 + \vec{A}^2 = \begin{cases} 0 & \text{bound states} \\ 1 & \text{asymptotically-free states} \end{cases} \quad (11.6)$$

Now let's take this conclusion as a rule: even in other situations, $|A_{\bar{\alpha}\beta}| = 0$ and 1 represent the two sides of boundary of **physical region**, beyond which there are singularities [Fig. 2]. As for a electron in hydrogen atom the singularity region (**nonlocal region**) is the region within the range of proton's wavelength.

As for the bound-state side of physical region, even in ElectroDynamics [10], one may encounter alternatives in which a particular approximation of $\vec{A} = 0$ seems unnecessary. For instance, that $1 - A_0^2 + \vec{A}^2 = 0$ resulting in $A_0^2 = 1 + \vec{A}^2$ looks also reasonable. However, in a perturbative theory there is no chance for any components of potential to have a value larger than 1. So in perturbative case, the only way to meet $1 - A_0^2 + \vec{A}^2 = 0$ is to make $\vec{A} = 0$ and $A_0 \rightarrow 1^-$. But the requirement to the asymptotically-free side is loose, because in any case the alternative solution $|A_0| \sim |\vec{A}|$ (additional to aforementioned $\vec{A} \rightarrow 0$, $A_0 \rightarrow 0$) of $1 - A_0^2 + \vec{A}^2 = 1$ is by no means forbidden.

B. Two Types of Curving in Colour Space

In section VI, we have noted that the metric form is relevant to the interaction vertex of CQFT. In QCD, we adopt the colour-spin-independent form to express its interaction vertex. Apart from a coupling constant, the vertex reads

$$\bar{\Psi}(x)\gamma_\mu A_a^\mu \lambda^a \Psi(x) = \bar{\Psi}(x)\gamma_\mu \lambda^a \Psi(x) A_a^\mu, \quad (11.7)$$

in which λ^a ($a = 1, \dots, 8$) are Gell-mann matrices. Correspondingly, the wave function for quarks can be also written in a separable form

$$\Psi(x) = \psi(x) \begin{pmatrix} q^a(x) \\ q^b(x) \\ q^c(x) \end{pmatrix}, \quad (11.8)$$

where $\psi(x)$ is the familiar spinor part [being transformed under the group $GL(4, \mathcal{C})$] and $\begin{pmatrix} q^a(x) \\ q^b(x) \\ q^c(x) \end{pmatrix}$ the colour part [being transformed under the group $U(3, \mathcal{C})$]. In this respect, the complete form of metric for colour interaction can be written

$$A(X, Y) = d\psi^\alpha(x) q_a(x) A_{\alpha\bar{\beta}}^{ab} q_b(x) d\bar{\psi}^\beta(x), \quad (11.9)$$

where the metric tensor $A_{\alpha\bar{\beta}}^{ab}$ may be further decomposed into

$$A_{\alpha\bar{\beta}}^{ab} = A_{\alpha\bar{\beta}} A^{ab} \text{ or } (A_{\alpha\bar{\beta}}^{ab})_{12 \times 12} = (A_{\alpha\bar{\beta}})_{4 \times 4} \otimes (A^{ab})_{3 \times 3}. \quad (11.10)$$

We have mentioned in section IV how the connection, curvature and the Ricci tensor are derived from $A_{\alpha\bar{\beta}}$. Being multiplied by an additional matrix element A^{ab} , the forms of the aforementioned geometry quantities constructed

from $A_{\alpha\bar{\beta}}$, particularly the Ricci tensor, will not be changed since we can make the indices a, b fixed temporarily so that the factor A^{ab} becomes a constant. The same procedure is equally applicable to fixing the indices $\alpha, \bar{\beta}$ and letting $(A^{ab})_{3\times 3}$ form the corresponding curvature and Ricci tensor. Combining the two procedures we obtain the Ricci tensor for the whole metric

$$R_{\bar{\rho}\sigma} = \frac{\partial^2 \ln |A_{\alpha\bar{\beta}}^{ab}|}{\partial \bar{z}^\rho \partial z^\sigma} = 3 \frac{\partial^2 \ln |A_{\alpha\bar{\beta}}|}{\partial \bar{\psi}^\rho \partial \psi^\sigma} + 4 \frac{\partial^2 \ln |A^{ab}|}{\partial \bar{q}^\rho \partial q^\sigma} = R_S + R^C, \quad (11.11)$$

where R_S is for spinor and R^C for colour. In view of section VII, we recognize that the metric tensor $A_{\alpha\bar{\beta}}$ is for the hyperbolic space and tensor A^{ab} is for ellipse colour space, i.e., R_S and R^C are subject to different geometries (different curvings) though their forms are similar. Consequently, the field equation $R_{\bar{\alpha}\beta} = 0$ now becomes

$$R_S = -R^C. \quad (11.12)$$

For simplicity in form, we assume that the constants or other variables before $\frac{\partial^2 \ln |A_{\alpha\bar{\beta}}|}{\partial \bar{\psi}^\rho \partial \psi^\sigma}$ and $\frac{\partial^2 \ln |A^{ab}|}{\partial \bar{q}^\rho \partial q^\sigma}$ have been absorbed in R_S and R^C .

The above equation provides us with a way to give rise to mass term alternatively, or reversely to eliminate the mass term in Eq. (9.29). In section-IX we have obtained the field equation in space-time $\vec{A} \cdot \square \vec{A} + (\vec{E}^2 - \vec{B}^2) = 0$, which can be derived from R_S in Eq.(11.12). On the other hand, $-R^C$ can contribute a nontrivial term to the field equation in space-time, and roughly we denote such a term by $m_C^2 \vec{A}^2$. So far the field equation for colour field is given by

$$\vec{A} \cdot \square \vec{A} + (\vec{E}^2 - \vec{B}^2) = m_C^2 \vec{A}^2. \quad (11.13)$$

From this equation one recognizes that even when $\vec{E}^2 - \vec{B}^2 = 0$ can a mass term be provided by $m_C^2 \vec{A}^2$, if only the dimension of boson field is larger than $1[U(1)]$. Possibly the term $m_C^2 \vec{A}^2$ can offer a mechanism to give rise to mass for gluons. But the reverse possibility exists simultaneously: if $\vec{E}^2 - \vec{B}^2 \neq 0$, the term $m_C^2 \vec{A}^2$ may happen to eliminate the contribution from $\vec{E}^2 - \vec{B}^2$ —when their signs are the same. So at the present formula-level it can't be asserted whether the gluon owns mass or not.

C. Physical Region in Colour Space and two Understandings to Colour Confinement

From both experiments and the theory of QCD, it has been well known that the colour interaction is asymptotically free, i.e. while the transferred momenta of quarks are very large, the coupling constant of colour interaction tends to zero and the interaction becomes a perturbative one. In this case, following the condition $1 - A_0^2 + \vec{A}^2 \rightarrow 1$ in Eq.(11.6), we have $A_0 \rightarrow 0$, $\vec{A} \rightarrow 0$ or $A_0 \sim \vec{A}$. The other side of boundary of quarks' physical region, where the quarks are tightly confined, is assumed to correspond to the case of bound electron in hydrogen atom. Now we begin to discuss this side of boundary of physical region for colour interaction.

When two quarks are in their ground state (a low-momentum state), according to Eq.(11.6) and the parallelism between confined quarks and bound electrons [Fig. 2], the condition $|A_{\alpha\bar{\beta}}^{ab}| = 0$ should hold. The Eq.(11.11) means that either $|A_{\alpha\bar{\beta}}| = 0$ or $|A^{ab}| = 0$ is satisfied. Among others, $|A^{ab}| = 0$ deserves more attention. Obviously it makes the rank of the colour matrix decrease at least by one, from $SU(3)$ to $SU(2)$ or $U(1)$, which suggests that quarks condense to hadrons, or possibly others, and thus no free quark appears. This conclusion seems able to account for the confinement of quarks, at least as a mechanism.

Next we examine the meaning of $|A_{\alpha\bar{\beta}}^{ab}| = 0$ in more details. Under our approximation (11.7), we have expressed the metric matrix as $(A_{\alpha\bar{\beta}}^{ab}) = (A_{\alpha\bar{\beta}})_{4\times 4} \otimes (A^{ab})_{3\times 3}$, in which the metric tensors of spinor part and colour part become separable. This separable form gives an understanding to colour confinement in the last paragraph. The understanding however, will not be abated even if the two parts of metric tensor get entangled, since anyway the condition $|A_{\alpha\bar{\beta}}^{ab}| = 0$ makes the rank of matrix $(A_{\alpha\bar{\beta}}^{ab})_{12\times 12}$ decrease at least by **one**, **which** belongs to either the spinor or colour space. Therefore in what follows we take into account the general matrix form with two parts entangled together. The following approximation of $A_{\alpha\bar{\beta}}^{ab}$ can be of an entangled form

$$(A_{\alpha\bar{\beta}}^{ab}) = \gamma^0 \otimes I_{3\times 3} + A_\mu \gamma^0 \gamma^\mu \otimes \vec{A}_a \lambda^a. \quad (11.14a)$$

The use of Eq. (11.14a) can free us from writing out the explicit space-time indices or colour indices in some special problems. For instance, in what follows we can only write \vec{A} to imply the inclusion of colour part, as $\vec{A} = \vec{A}_a \lambda^a$,

$$(A_{\alpha\beta}^{ab}) = \gamma^0 + A_\mu \gamma^0 \gamma^\mu. \quad (11.14b)$$

For a perturbative interaction, it is impossible for scalar potential and vector potential to satisfy the relation $|A_{\alpha\beta}| = 1 - A_0^2 + \vec{A}^2 = 0$ with $\vec{A}^2 > 0$. To avoid the coincidence with perturbative interaction, for colour interaction we prefer the condition $\vec{A}^2 > 0$ rather than $\vec{A} \sim 0$, $A_0 \sim 1$ to give the solution of $A_0^2 = 1 + \vec{A}^2$. So we turn to the case $\vec{A}^2 > 0$ and $A_0^2 > 1$. We recognize that this case doesn't rule out the possibility that while keeping the difference $A_0^2 - \vec{A}^2$ equal to 1, we can at the same time make both \vec{A} and A_0 continuously decrease until A_0 approaches the *perturbative* situation $A_0 \sim 1$ with $\vec{A}^2 \sim \varepsilon$ (infinitesimal value). Therefore $\vec{A}^2 > 0$ seems also coincident with *perturbative* situation, then we have to consider $\vec{A}^2 < 0$. Without affecting the main conclusion, let's assume that the limiting situation $A_0^2 = 0$ holds. Under this situation, we have $\vec{A} = \pm i \vec{A}_W$ (\vec{A}_W is the corresponding real vector).

Now let's take into account the explicit form $\vec{A} = \vec{A}_a \lambda^a$. Since every component of \vec{A} is a matrix, for example $A_3 = A_a^3 \lambda^a$, the value of any component should be one of the eigenvalues (of the corresponding matrix) or their combination (the coefficients A_a^3 , $a = 1, \dots, 8$, are real) after tracing out the degree of freedom of colour interaction by, e.g.

$$(q_f^A \ q_f^B \ q_f^C) A_a^3 \lambda^a \begin{pmatrix} q_i^a \\ q_i^b \\ q_i^c \end{pmatrix}, \quad (11.15)$$

where the subscript i and f represent initial and final states respectively. It can be asserted all values of components are real because the generators λ^a ($a = 1, \dots, 8$) of group $SU(3)$ are hermitian (we choose Gellmann matrices as the generators). Then it appears to be inconsistent with the previous conclusion $\vec{A} = \pm i \vec{A}_W$ which is a pure imaginary vector. One way to treat the inconsistency is to extend the 8-generators of $SU(3)$ to 9-generators of $U(3)$

$$A_3 = A_0^3 i I_{3 \times 3} + A_a^3 \lambda^a, \quad a = 1, \dots, 8, \quad (11.16)$$

in which an additional pure imaginary generator $i I_{3 \times 3}$ is introduced, we denote it as λ^0 . We find λ^0 , with $\lambda^0 (\lambda^0)^\dagger = 1$, is not a Hermitian matrix, so that its eigenvalues are not real. Now we conclude that when the confinement happens only the first coefficient \vec{A}_0 is nontrivial, it yields

$$\vec{A} = i(A_0^1, A_0^2, A_0^3). \quad (11.17)$$

On condition that $A_0^2 \neq 0$, the same extension of "8-generators to 9-generators" would be necessary, and \vec{A} is a complex number accordingly.

The above extension of generators is reasonable because if a space is curved, then some extra dimensions begin to get involved. For instance, after a plane is curved, one must choose three-dimension space and the corresponding transformation group in it to describe the curving. We choose $U(3)$ space as the curved quark space in order to satisfy the confining condition.

In summary, in this subsection the equation $|A_{\alpha\beta}^{ab}| = 0$ leads us to two explanations of the colour confinement: First, it gives the decrease of the rank of colour matrix, which is a general conclusion independent of separable-approximation. Second, it induces the curving of colour space, which is a conclusion dependent on the approximation (11.14) and (9.10).

XII. SUMMARY AND DISCUSSION

Relationship with conventional Quantum Field Theory (CQFT) The geometry model presented in this paper can also be viewed as a theory for elemental quantum fields, but it lacks many characteristics of CQFT. As for symmetries, the transformations responsible for the groups $GL(n, \mathcal{C})$ or $U(n, \mathcal{C})$ are used in this paper. Whereas the concept of gauge transformation, under which the invariance of Lagrangian or Action is respected in CQFT, is assumed to be irrelevant here. So the conservation laws due to invariant Lagrangian are not appropriate to discuss in this frame. However, from purely geometrical angle, we respect the invariance of geometric quantities such as metric and curvature etc. under the transformation of structure group. Generally speaking, in a nonlocal theory any conservation law holds only when some integrations are carried out over certain spaces, and thus the square root of Jaccobian should appear in the forms of conservation law, as that in GR and implied in Sections 8~10.

Furthermore, generally in CQFT the Lagrangian depends on the fields and their first derivatives only. But in differential geometric, it allows for up to two derivatives of fields, which can be noticed in the quadratic form of motion equation for fermions and field equation for bosons.

In view of the distinguished success of CQFT in describing the purterbative interactions, the results from these sectors are respected. First, the Dirac equation and Klein-Gordon equation in QED, the asymptotically freedom in QCD etc. are employed to compare with the results of our model. In that respect, the quadratic forms of Dirac equation and Klein-Gordon equation are regained under some approximations. Second, the interaction vertices of QED and QCD are used to design appropriate approximations of metric tensor used in the second half of the paper. However, the technique of renormalization is assumed unnecessary in this theory. Because the inclusion of the nonlocal characteristics of quantum wave in this theory would automatically make the value of particles' momentum within a limited range.

Relationship with General Relativity (GR) As stated in the introduction, the formalism developed in this work is similar to that of GR. The difference is that we have generalized the space from real to complex. The important characteristic of our model is that the base manifold is complex. If the space-time is initially used as the base manifold, it is impossible to clarify physical meanings of dynamics or to get the results of this paper. Additionally, the manifolds under concern are not necessarily Riemann manifold, i.e. the metric tensor is not necessarily symmetric, $A(X, Y) = A(Y, X)$.

In GR, only one type of curving—Riemann curving (hyperbolic curving)—appears, but in our theory two curvings are present—both hyperbolic and elliptic curvings. They are represented by two different structure groups $GL(n, \mathcal{C})$ and $U(n, \mathcal{C})$, which correspond to two differential geometries. The geometry with structure group $U(n, \mathcal{C})$ is relevant to Riemann manifold [e.g. $1/2(A(X, Y) + A(Y, X))$ gives Riemann metric], but $GL(n, \mathcal{C})$ not. In non-Abelian case, the metric tensors from two geometries may be entangled as shown in (11.14).

The Approximations Used in This Paper All the meaningful results are dependent on the four approximations in Eqs. (9.10), (10.13), (11.10) and (11.14). The credibility of the results relies on the reasonability of the approximations. The approximations follow from our understanding of the interaction vertex in CQFT. The appreciated aspect of these approximations is the restoration of the terms of quadratic Dirac equation and Klein-Gordon equation by the aids of the replacement $d \rightarrow \gamma_\mu \partial^\mu$. On the other hand, the appearance of mass term for bosons and the understanding to quark confinement also follow from above approximations and the replacement.

From complex space to real space we use $d \rightarrow \gamma_\mu \partial^\mu$ as a projection, we have to do so to get some physical observables in space-time. In doing so some information is inevitably lost, but it cannot be avoided—the observed space is complex, the observing space is real. Strictly speaking, the replacement $d \rightarrow \gamma_\mu \partial^\mu$ is not an approximation, but rather a technique following the approximation $(A^{\alpha\bar{\rho}})_{4 \times 4} = \gamma_0 - \gamma_0 \gamma^\mu A_\mu$. It helps us gain the form of motion equation for fermions, though with quite a few additional integrations involved. To resolve that intricate motion equation possibly requires us to convert it back to the Dirac-equation form by using some special techniques.

Remarks on applicability of the theory Although the model is designed for a low-energy nonperturbative interaction, at present almost all the approximations imposed on the model have perturbative forms and originations. And thus all the results here rely on both the theory and the approximations. We have noticed some results, such as the additional terms in motion equation and the meaning of $|A_{\alpha\beta}^{ab}| = 0$ etc., do not completely fit to the perturbative cases, which is expected to be the nonperturbative signals of our model. So after the model is reduced to get the main essences of CQFT, we hope it can be perfectly useful for nonperturbative interaction [e. g. in phenomenological study of the light-quark excited states.] or, long-wavelength low-energy region of perturbative interaction [e. g. strong correlations of fermions in ultra low temperature.], as anticipated initially.

Future Development of the theory First, we should search for a smaller group than $GL(4, \mathcal{C})$ to accurately describe the hyperbolic curving (Lorentz violation of spinors' transformation), as stated in Sec. 7. The search progress may involve some new physics, not a merely mathematical problem. Second, we should present a more reliable way to give the mass term in fermion motion equation in Sec. 8. C, some numerical calculation may be involved. Third, we hope the theory applicable to weak interaction too. Relevantly, the final understanding of mass problem relies undoubtedly on the issues of weak interaction.

Acknowledgments

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XIII. APPENDIX. A EVALUATING THE INTEGRAL OF EQ. (8.10)

In this appendix we give the detailed steps and methods of evaluating the first integration in Eq. (8.10).

$$\begin{aligned}
& \int (1 - \gamma_\mu A^\mu) \gamma_\sigma \partial^\sigma (1 + \gamma_\nu A^\nu) \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda \\
&= \int \gamma_\sigma \partial^\sigma (\gamma_\nu A^\nu) \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda \\
&\quad - \int \gamma_\mu A^\mu \gamma_\sigma \partial^\sigma (\gamma_\nu A^\nu) \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda, \quad (8.10)
\end{aligned}$$

here $\tilde{\psi} = \{\psi^1, \psi^2, \psi^3, \psi^4\}$ is a four-component vector, $A^{\alpha\bar{\rho}}$ and $A_{\bar{\rho}\gamma}$ are all in their matrix forms, $1 - \gamma_\mu A^\mu$ and $1 + \gamma_\nu A^\nu$. We will mainly deal with the first integration in (8.10), since the second term includes two A^μ factors we temporarily omit it as second-order perturbation.

$$\begin{aligned}
& \int \gamma_\sigma \partial^\sigma (\gamma_\nu A^\nu) \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda \\
&= \int \gamma_\sigma \gamma_\nu (\partial^\sigma A^\nu) \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda \\
&= \int (g_{\sigma\nu} + \underline{\gamma_\sigma \gamma_\nu}) (\partial^\sigma A^\nu) \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda \\
&= \int \partial_\nu A^\nu \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda - \int \partial^\sigma (\gamma_\nu A^\nu) \gamma_\sigma \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda \quad (A.1)
\end{aligned}$$

henceforth in this Appendix we use $\underline{\gamma_\sigma \gamma_\nu}$ to express the product $\gamma_\sigma \gamma_\nu$ when $\sigma \neq \nu$. We evaluate the above two terms in (A.1) separately. The first term

$$\begin{aligned}
& \int \partial_\nu A^\nu \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda \\
&= \int \partial_\nu A^\nu (g_{\rho\lambda} + \underline{\gamma_\rho \gamma_\lambda}) \partial^\rho \psi dx^\lambda \\
&= \int \partial_\nu A^\nu \partial_\lambda \psi dx^\lambda + \frac{1}{2} \gamma_\rho \gamma_\lambda \int \partial_\nu A^\nu (dx^\lambda \partial^\rho - dx^\rho \partial^\lambda) \psi \\
&= \partial_\nu A^\nu \psi \big|_{-\infty}^t - \int \psi (\partial_\lambda \partial_\nu A^\nu) dx^\lambda + \frac{1}{2} \gamma_\rho \gamma_\lambda \int \partial_\nu A^\nu (dx^\lambda \partial^\rho - dx^\rho \partial^\lambda) \psi \quad (A.2)
\end{aligned}$$

To evaluate the second term in (A.1), the following relation is useful:

$$\gamma_\sigma \gamma_\rho \gamma_\lambda = \begin{cases} -i \epsilon_{\sigma\rho\lambda\mu} \gamma_5 \gamma^\mu, & \text{if } \sigma, \rho, \lambda \text{ unequal to each other,} \\ g_{\sigma\rho} \gamma_\lambda + g_{\rho\lambda} \gamma_\sigma - g_{\sigma\lambda} \gamma_\rho, & \text{if at least two of } \sigma, \rho, \lambda \text{ equal to each other.} \end{cases} \quad (A.3)$$

Substituting this relation to second term in (A.1) yields,

$$\begin{aligned}
& \int \partial^\sigma (\gamma_\nu A^\nu) \gamma_\sigma \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda \\
&= -i \epsilon_{\sigma\rho\lambda\mu} \gamma_5 \gamma^\mu \int \partial^\sigma (\gamma_\nu A^\nu) \partial^\rho \psi dx^\lambda + \\
&\quad \int \partial^\sigma (\gamma_\nu A^\nu) (g_{\sigma\rho} \gamma_\lambda + g_{\rho\lambda} \gamma_\sigma - g_{\sigma\lambda} \gamma_\rho) \partial^\rho \psi dx^\lambda \quad (A.4)
\end{aligned}$$

We will leave the first term of (A. 4) as it is and turn to cope with the second term,

$$\begin{aligned}
& \int \partial^\sigma (\gamma_\nu A^\nu) (g_{\sigma\rho} \gamma_\lambda + g_{\rho\lambda} \gamma_\sigma - g_{\sigma\lambda} \gamma_\rho) \partial^\rho \psi dx^\lambda \\
&= \int \partial^\sigma A^\nu \gamma_\nu \gamma_\lambda \partial_\sigma \psi dx^\lambda + \int \partial^\sigma A^\nu \gamma_\nu \gamma_\sigma \partial^\lambda \psi dx_\lambda + \int \partial^\lambda A^\nu \gamma_\nu \gamma_\rho \partial^\rho \psi dx_\lambda \\
&= \int \partial^\sigma A^\nu (g_{\nu\lambda} + \underline{\gamma_\nu \gamma_\lambda}) \partial_\sigma \psi dx^\lambda + \int \partial^\sigma A^\nu (g_{\nu\sigma} + \underline{\gamma_\nu \gamma_\sigma}) \partial^\lambda \psi dx_\lambda \\
&\quad + \int \partial^\lambda A^\nu (g_{\nu\rho} + \underline{\gamma_\nu \gamma_\rho}) \partial^\rho \psi dx_\lambda \\
&= \int \partial^\sigma A^\nu \partial_\sigma \psi dx_\nu + \frac{1}{2} \gamma_\nu \gamma_\lambda \int (dx^\lambda \partial^\sigma A^\nu - dx^\nu \partial^\sigma A^\lambda) \partial_\sigma \psi \\
&\quad + \int \partial_\nu A^\nu \partial^\lambda \psi dx_\lambda + \frac{1}{2} \gamma_\nu \gamma_\sigma \int F^{\sigma\nu} \partial^\lambda \psi dx_\lambda \\
&\quad + \int \partial^\lambda A_\nu \partial^\nu \psi dx_\lambda - \frac{1}{2} \gamma_\nu \gamma_\rho \int (\partial^\lambda A^\nu \partial^\rho - \partial^\lambda A^\rho \partial^\nu) \psi dx_\lambda \quad (A.5)
\end{aligned}$$

where $F^{\sigma\nu} = (\partial^\sigma A^\nu - \partial^\nu A^\sigma)$. Then performing the integration by parts, it yields

$$\begin{aligned}
& \int \partial^\sigma (\gamma_\nu A^\nu) (g_{\sigma\rho} \gamma_\lambda + g_{\rho\lambda} \gamma_\sigma - g_{\sigma\lambda} \gamma_\rho) \partial^\rho \psi dx^\lambda \\
&= \gamma_\nu \gamma_\lambda \int \partial^\sigma A^\nu \partial_\sigma \psi dx^\lambda + \\
&(\partial_\nu A^\nu) \psi \Big|_{-\infty}^t - \int \psi (\partial_\lambda \partial_\nu A^\nu) dx^\lambda - \frac{1}{2} \gamma_\nu \gamma_\sigma F^{\nu\sigma} \psi \Big|_{-\infty}^t + \\
&\quad \frac{1}{2} \gamma_\nu \gamma_\sigma \int \psi (\partial_\lambda F^{\nu\sigma}) dx^\lambda - \\
&A^\nu \partial_\nu \psi \Big|_{-\infty}^t + \int A^\nu (\partial_\lambda \partial_\nu \psi) dx^\lambda - \frac{1}{2} \gamma_\nu \gamma_\rho \tilde{F}^{\nu\rho} \psi \Big|_{-\infty}^t + \\
&\quad \frac{1}{2} \gamma_\nu \gamma_\rho \int \tilde{F}^{\nu\rho} \partial_\lambda \psi dx^\lambda, \quad (A.6)
\end{aligned}$$

where $\tilde{F}^{\mu\nu} = A^\mu \partial^\nu - A^\nu \partial^\mu$. In the first term of the right hand we have converted the expansion $\int \partial^\sigma A^\nu \partial_\sigma \psi dx_\nu + \frac{1}{2} \gamma_\nu \gamma_\lambda \int (dx^\lambda \partial^\sigma A^\nu - dx^\nu \partial^\sigma A^\lambda) \partial_\sigma \psi$ back to its original form for there is no way integrating out the coordinates to simple form. Now substituting Eq. (A.6) and Eq. (A.2) into Eq. (A.1) leads to

$$\begin{aligned}
& \int \gamma_\sigma \partial^\sigma (\gamma_\nu A^\nu) \gamma_\rho \gamma_\lambda \partial^\rho \psi dx^\lambda \\
&= A_\nu \partial^\nu \psi + \frac{1}{2} \gamma_\mu \gamma_\nu \tilde{F}^{\mu\nu} \psi + \frac{1}{2} \gamma_\mu \gamma_\nu F^{\mu\nu} \psi \\
&\quad + \frac{1}{2} \gamma_\rho \gamma_\lambda \int (\partial_\nu A^\nu) (dx^\lambda \partial^\rho - dx^\rho \partial^\lambda) \psi \\
&\quad + i \epsilon_{\sigma\rho\lambda\mu} \gamma_5 \gamma^\mu \int \partial^\sigma (\gamma_\nu A^\nu) \partial^\rho \psi dx^\lambda - \int A_\nu \partial_\lambda \partial^\nu \psi dx^\lambda - \frac{1}{2} \gamma_\mu \gamma_\nu \int \tilde{F}^{\mu\nu} \partial_\lambda \psi dx^\lambda \\
&\quad - \frac{1}{2} \gamma_\mu \gamma_\nu \int \partial_\lambda F^{\mu\nu} \psi dx^\lambda - \gamma_\mu \gamma_\nu \int \partial^\sigma A^\mu \partial_\sigma \psi dx^\nu. \quad (A.7)
\end{aligned}$$

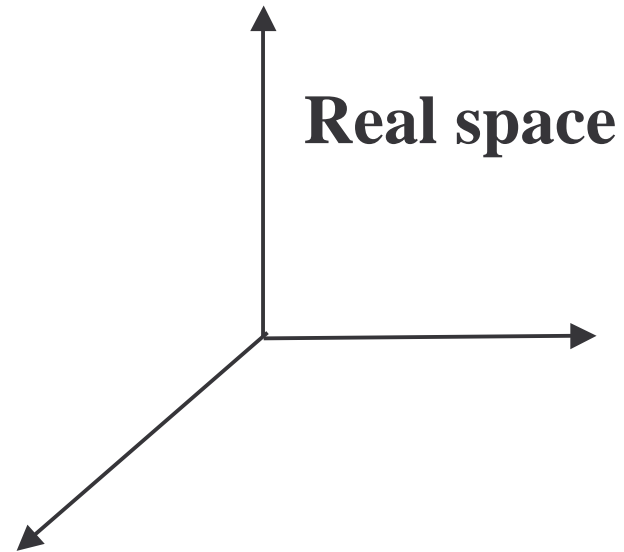
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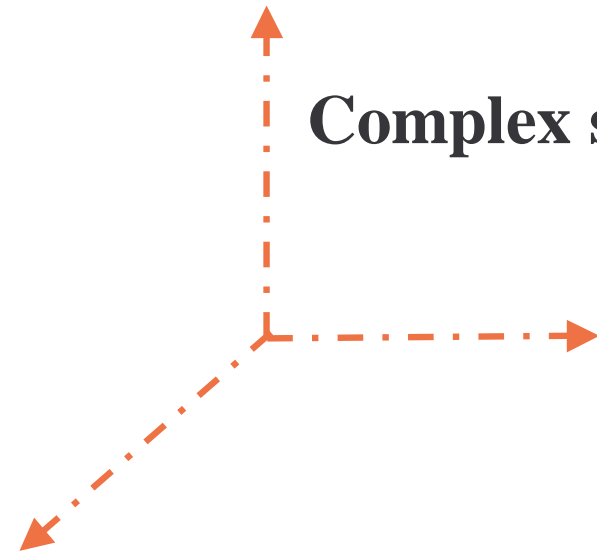
Figure Captions

Fig. 1: To describe how one fermion observes (interacts with) another fermion, we employ the formalism of General Relativity by generalizing its space from real to complex. In graph, only three axes are displayed, in fact we work in four-dimension space.

Fig. 2: Patterns of physical region in three-dimension space, respectively for electrons (A) and quarks (B). Likewise the singularity and physical region for complex space are defined in Sec. 11 by adding constraints to interaction potential.

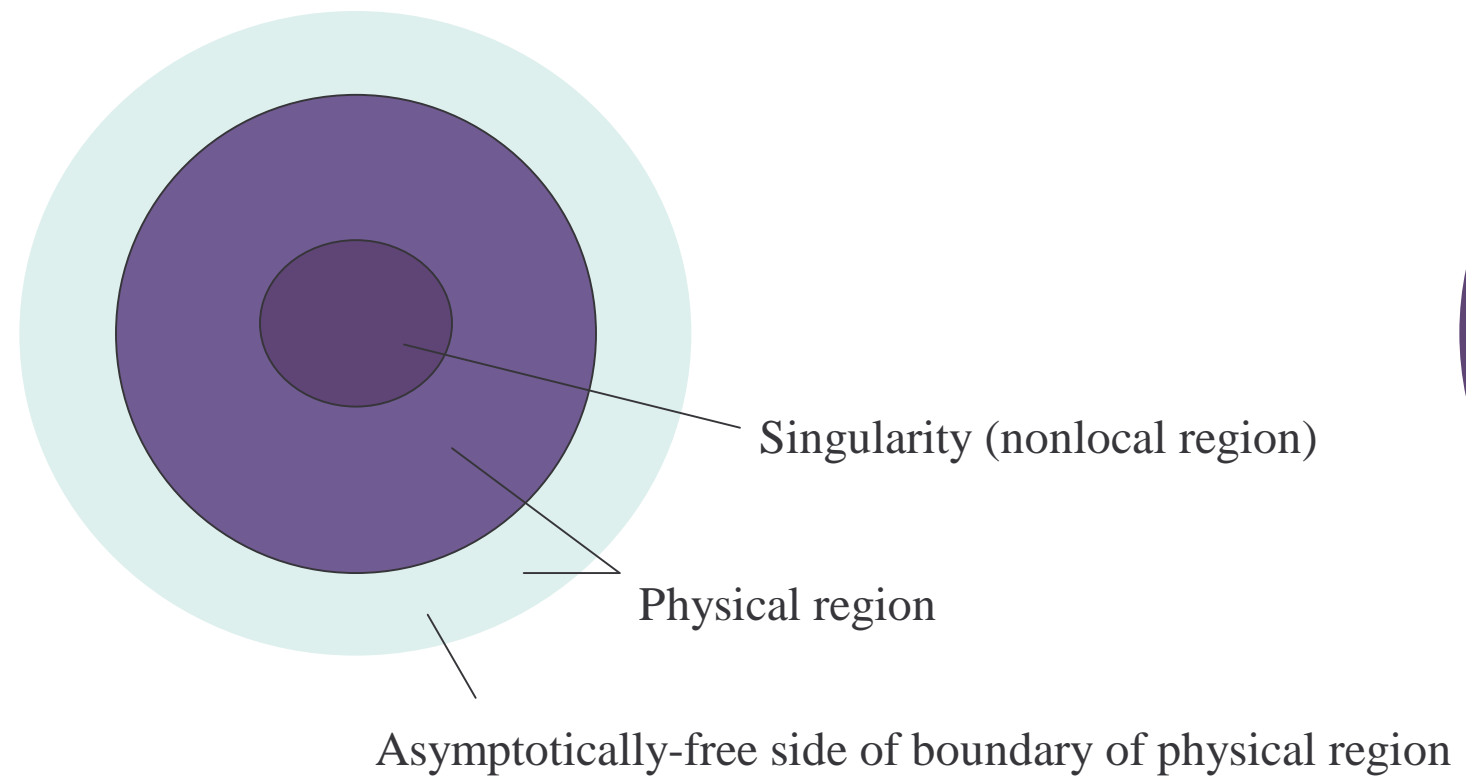


Real space

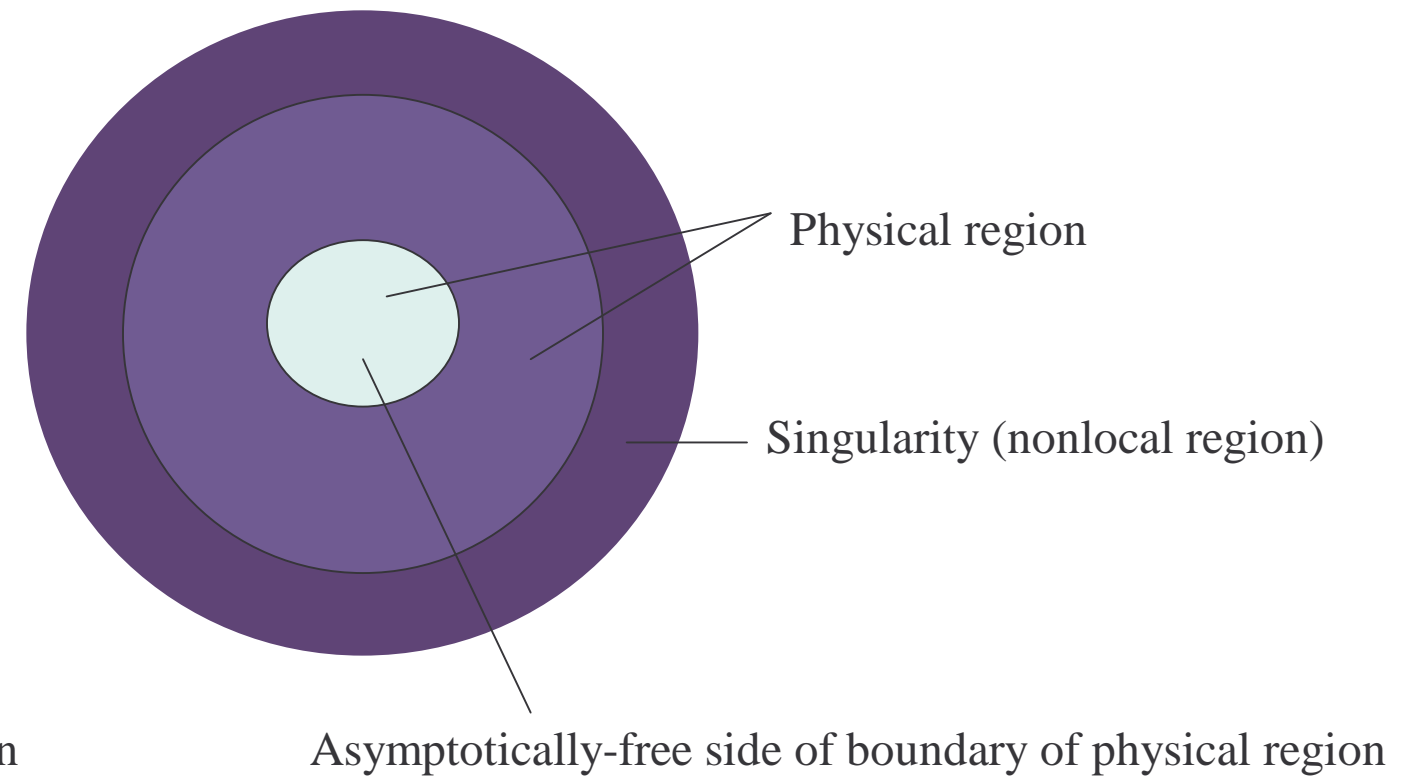


Complex space

Each of axes is replaced by a complex one.



(A)



(B)